Strongly correlated many-body systems: from electronic materials to ultracold atoms to photons

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Thanks to:

“Conventional” solid state materials

Bloch theorem for non-interacting electrons in a periodic potential

Electronic band structure of GaAs
“Conventional” solid state materials

Electron-phonon and electron-electron interactions are irrelevant at low temperatures.

\[ \frac{1}{\tau_{e-e}} \sim \epsilon^2 \quad \frac{1}{\tau_{e-ph}} \sim \epsilon^3 \]

Landau Fermi liquid theory: when frequency and temperature are smaller than \( E_F \) electron systems are equivalent to systems of non-interacting fermions.

\[ \rho = \rho_0 + a T^2 \]

\[ c/T = \text{const} \]

\[ \kappa/T = \text{const} \]
Strongly correlated electron systems

Quantum Hall systems
kinetic energy suppressed by magnetic field

One dimensional electron systems
non-perturbative effects of interactions in 1d

High temperature superconductors,
Heavy fermion materials,
Organic conductors and superconductors
many puzzling non-Fermi liquid properties
Bose-Einstein condensation of weakly interacting atoms

Scattering length is much smaller than characteristic interparticle distances. Interactions are weak

\[ n \sim 10^{14} \text{ cm}^{-3} \quad \text{and} \quad T_{\text{BEC}} \sim 1 \mu\text{K} \]
Strongly correlated systems of cold atoms

- Optical lattices
- Feshbach resonances
- Low dimensional systems
Linear geometrical optics

Newton’s experiment for splitting white light into a spectrum
Strongly correlated systems of photons

Strongly interacting polaritons in coupled arrays of cavities
M. Hartmann et al., Nature Physics (2006)

Strong optical nonlinearities in nanoscale surface plasmons

Crystallization (fermionization) of photons in one dimensional optical waveguides
D. Chang et al., arXive:0712.1817
We understand well: many-body systems of non-interacting or weakly interacting particles. For example, electron systems in semiconductors and simple metals. When the interaction energy is smaller than the kinetic energy, perturbation theory works well.

We do not understand: many-body systems with strong interactions and correlations. For example, electron systems in novel materials such as high temperature superconductors. When the interaction energy is comparable or larger than the kinetic energy, perturbation theory breaks down. Many surprising new phenomena occur, including unconventional superconductivity, magnetism, fractionalization of excitations.
Ultracold atoms have energy scales of $10^{-6}$K, compared to $10^4$ K for electron systems.

By engineering and studying strongly interacting systems of cold atoms we should get insights into the mysterious properties of novel quantum materials.

We will also get new systems useful for applications in quantum information and communications, high precision spectroscopy, metrology.
Strongly interacting systems of ultracold atoms and photons:

**NOT the analogue simulators**

These are independent physical systems with their own “personalities”, physical properties, and theoretical challenges.
Focus of these lectures: challenges of new strongly correlated systems

Part I
Detection and characterization of many body states
Quantum noise analysis and interference experiments

Part II
New challenges in quantum many-body theory: non-equilibrium coherent dynamics
Quantum noise studies of ultracold atoms
Outline of part I

Introduction. Historical review

Quantum noise analysis of the time of flight experiments with ultracold atoms (HBT correlations and beyond)

Quantum noise in interference experiments with independent condensates

Quantum noise analysis of spin systems
Quantum noise

Classical measurement:
collapse of the wavefunction into eigenstates of $x$

$$\langle x \rangle = \int dx \, x \, |\psi(x)|^2$$

$$\langle x^2 \rangle = \int dx \, x^2 \, |\psi(x)|^2$$

$$\ldots$$

$$\langle x^n \rangle = \int dx \, x^n \, |\psi(x)|^2$$

$$\ldots$$

Histogram of measurements of $x$
Probabilistic nature of quantum mechanics

Bohr-Einstein debate on spooky action at a distance

Einstein-Podolsky-Rosen experiment

\[ |S = 0 \rangle = | \uparrow \rangle_L | \downarrow \rangle_R - | \downarrow \rangle_L | \uparrow \rangle_R \]

Measuring spin of a particle in the left detector instantaneously determines its value in the right detector
Aspect’s experiments: tests of Bell’s inequalities

Correlation function

\[ E(\theta_1, \theta_2) = \frac{\langle (I_1^+ - I_1^-)(I_2^+ - I_2^-) \rangle}{\langle (I_1^+ + I_1^-)(I_2^+ + I_2^-) \rangle} \]

Classical theories with hidden variable require

\[ B = E(\theta_1, \theta_2) - E(\theta_1, \theta'_2) + E(\theta'_1, \theta'_2) - E(\theta'_1, \theta_2) \leq 2 \]

Quantum mechanics predicts \(B=2.7\) for the appropriate choice of \(\theta\)'s and the state

\[ |\psi\rangle = |+\rangle_L |+\rangle_R + |-\rangle_L |-\rangle_R \]

Hanbury-Brown-Twiss experiments

Classical theory of the second order coherence

\[ \langle I(\vec{r}_1) I(\vec{r}_2) \rangle = A + B \cos \left( (\vec{k} - \vec{k}') (\vec{r}_1 - \vec{r}_2) \right) \]

Measurements of the angular diameter of Sirius


Shot noise in electron transport

Proposed by Schottky to measure the electron charge in 1918

Spectral density of the current noise

\[ S_\omega = \int \langle \{ \delta I(t), \delta I(0) \} \rangle e^{i\omega t} dt \]

Related to variance of transmitted charge

\[ S_0 = \frac{2}{\tau} \langle \delta q^2(\tau) \rangle \]

When shot noise dominates over thermal noise

\[ S_0 = 2 e I \]

Poisson process of independent transmission of electrons
Shot noise in electron transport

Current noise for tunneling across a Hall bar on the 1/3 plateau of FQE

Etien et al. PRL 79:2526 (1997)
see also Heiblum et al. Nature (1997)
Quantum noise analysis of the time of flight experiments with ultracold atoms (HBT correlations and beyond)


Experiments: Folling et al., Nature 434:481 (2005);
              Greiner et al., PRL 94:110401 (2005);
              Tom et al. Nature 444:733 (2006);

see also    Hadzibabic et al., PRL 93:180403 (2004)
            Spielman et al., PRL 98:80404 (2007);
            Guerrera et al., preprint (2007)
Atoms in optical lattices

Theory: Jaksch et al. PRL (1998)

Experiment: Kasevich et al., Science (2001);
Greiner et al., Nature (2001);
Esslinger et al., PRL (2004);
Ketterle et al., PRL (2006)
Bose Hubbard model

\[ \mathcal{H} = -t \sum_{\langle ij \rangle} b_i^+ b_j + U \sum_i n_i (n_i - 1) - \mu \sum_i n_i \]

- \( t \) — tunneling of atoms between neighboring wells
- \( U \) — repulsion of atoms sitting in the same well
Superfluid to insulator transition in an optical lattice

Why study ultracold atoms in optical lattices?
Fermionic atoms in optical lattices

\[ \mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i} n_{i} \]

Experiments with fermions in optical lattice, Kohl et al., PRL 2005
YBa$_2$Cu$_3$O$_7$

Antiferromagnetic and superconducting Tc of the order of 100 K

Atoms in optical lattice

Antiferromagnetism and pairing at sub-micro Kelvin temperatures

Same microscopic model

$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_i n_i$$
Positive U Hubbard model


Antiferromagnetic insulator

D-wave superconductor
Atoms in optical lattice

YBa$_2$Cu$_3$O$_7$

Same microscopic model

Quantum simulations of strongly correlated electron systems using ultracold atoms

Detection?
Quantum noise analysis as a probe of many-body states of ultracold atoms
Time of flight experiments

Quantum noise interferometry of atoms in an optical lattice

Second order coherence

$$G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle$$
Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment

Hanbury-Brown-Twiss stellar interferometer

\[ \langle I(\vec{r}_1) I(\vec{r}_2) \rangle = A + B \cos \left( (\vec{k} - \vec{k}') (\vec{r}_1 - \vec{r}_2) \right) \]
Hanbury-Brown-Twiss interferometer

\[
E(\vec{r}_1) = E_k e^{i\vec{k}\vec{r}_1} + E_{k'} e^{i\vec{k}'\vec{r}_1}
\]

\[
E(\vec{r}_2) = E_k e^{i\vec{k}\vec{r}_2} + E_{k'} e^{i\vec{k}'\vec{r}_2}
\]

\[
\langle I(r_1)I(r_2) \rangle = \langle |r_1|^2 |r_2|^2 \rangle
= \left\langle \left\{ |E_k|^2 + |E_{k'}|^2 + (E_k E_{k'}^* e^{i(\vec{k}-\vec{k}')\vec{r}_1} + \text{c.c.}) \right\} \right\rangle \left\{ |E_k|^2 + |E_{k'}|^2 + (E_k E_{k'}^* e^{i(\vec{k}-\vec{k}')\vec{r}_2} + \text{c.c.}) \right\}
= \langle (|E_k|^2 + |E_{k'}|^2)^2 \rangle + \langle |E_k|^2 |E_{k'}|^2 \rangle [e^{i(\vec{k}-\vec{k}')\vec{r}_1} + \text{c.c.}]
Quantum theory of HBT experiments

Glauber, *Quantum Optics and Electronics* (1965)

HBT experiments with matter

For bosons

\[ A = A_1 + A_2 \]

For fermions

\[ A = A_1 - A_2 \]

Experiments with neutrons

Experiments with electrons

Experiments with 4He, 3He
Westbrook et al., Nature (2007)

Experiments with ultracold atoms
Second order coherence in the insulating state of bosons.
Hanbury-Brown-Twiss experiment

Second order coherence in the insulating state of bosons

Bosons at quasimomentum $\vec{k}$ expand as plane waves with wavevectors $\vec{k}, \vec{k} + \vec{G}_1, \vec{k} + \vec{G}_2$

First order coherence: $\langle \rho(\vec{r}) \rangle$

Oscillations in density disappear after summing over $\vec{k}$

Second order coherence: $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle$

Correlation function acquires oscillations at reciprocal lattice vectors

$$\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle = A_0 + A_1 \cos \left( \vec{G}_1(\vec{r}_1 - \vec{r}_2) \right) + A_2 \cos \left( \vec{G}_2(\vec{r}_1 - \vec{r}_2) \right) + \ldots$$
Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment

Interference of an array of independent condensates


Smooth structure is a result of finite experimental resolution (filtering)
Quantum theory of HBT experiments

For bosons

\[ A = A_1 + A_2 \]

For fermions

\[ A = A_1 - A_2 \]
Second order coherence in the insulating state of fermions. Hanbury-Brown-Twiss experiment

How to detect antiferromagnetism
Probing spin order in optical lattices

Correlation Function Measurements

\[ G(r_1, r_2) = \langle n(r_1) \ n(r_2) \rangle_{\text{TOF}} - \langle n(r_1) \rangle_{\text{TOF}} \langle n(r_2) \rangle_{\text{TOF}} \]
\[ \sim \langle n(k_1) \ n(k_2) \rangle_{\text{LAT}} - \langle n(k_1) \rangle_{\text{LAT}} \langle n(k_2) \rangle_{\text{LAT}} \]

Extra Bragg peaks appear in the second order correlation function in the AF phase.
How to detect fermion pairing

Quantum noise analysis of TOF images: beyond HBT interference
Second order interference from the BCS superfluid

\[ \Delta n(r, r') = \Delta n(r) - \Delta n(r') \]

\[ \Delta n(r, -r) | \Psi_{BCS} \rangle = 0 \]
Momentum correlations in paired fermions

Greiner et al., PRL 94:110401 (2005)
Fermion pairing in an optical lattice

Second Order Interference
In the TOF images

\[ G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle \]

Normal State

\[ G_N(r_1, r_2) = \delta(r_1 - r_2)\rho(r_1) - \rho^2(r_1) \sum_G \delta(r_1 - r_2 - \frac{G\hbar t}{m}) \]

Superfluid State

\[ G_S(r_1, r_2) = G_N(r_1, r_2) + \Psi(r_1) \sum_G \delta(r_1 + r_2 + \frac{G\hbar t}{m}) \]

\[ \Psi(r) = |u(Q(r))v(Q(r))|^2 \] measures the Cooper pair wavefunction

One can identify unconventional pairing

\[ Q(r) = \frac{mr}{\hbar t} \]
How to see a “cat” state in the collapse and revival experiments

Quantum noise analysis of TOF images: beyond HBT interference
Collapse and revival experiments with bosons in an optical lattice

Initial superfluid state

Coherent state in each well at \( t=0 \)

\[
|\alpha\rangle = e^{-|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\]

Increase the height of the optical lattice abruptly

Individual wells isolated

Time evolution within each well

\[
\mathcal{H} = \frac{U}{2} n(n-1)
\]

\[
|n(t)\rangle = |n\rangle e^{-iUn(n-1)t/2\hbar}
\]
Collapse and revival experiments with bosons in an optical lattice

The collapse occurs due to the loss of coherence between different number states

\[ | n(t) \rangle = | n \rangle e^{-iUn(n-1)t/2\hbar} \]

At revival times \( t_r = \hbar/U, 2\hbar/U, \ldots \) all number states are in phase again.
Collapse and revival experiments with bosons in an optical lattice


Dynamical evolution of the interference pattern after jumping the optical lattice potential

![Image of interference patterns](image-url)
Collapse and revival experiments with bosons in an optical lattice


Quantum dynamics of a coherent states $|\langle \beta | \psi(t) \rangle |^2$

Cat state at $t=t_r/2$

$| \text{"cat"} \rangle = |\tilde{\alpha} \rangle + | -\tilde{\alpha} \rangle$
How to see a “cat” state in collapse and revival experiments

Romero-Isart et al., unpublished

Properties of the “cat” state:
no first order coherence

\[ \langle \psi(t_r/2) | a \gamma \psi(t_r/2) \rangle = 0 \]

pairing-like correlations

\[ \langle \psi(t_r/2) | a^2 | \psi(t_r/2) \rangle \neq 0 \]

In the time of flight experiments this should lead to correlations between \( n_{+k} \) and \( n_{-k} \)

Define correlation function (overlap original and flipped images)

\[ C(d) = \frac{\int \langle \hat{n}_{q+d/2} \hat{n}_{-q+d/2} \rangle dq}{\int \langle \hat{n}_{q+d/2} \rangle \langle \hat{n}_{-q+d/2} \rangle dq} \]
How to see a “cat” state in collapse and revival experiments

Exact diagonalization of the 1d lattice system. $U/t=3$, $N=2$

\[ C(d) = \frac{\int \langle \hat{n}_{q+d/2} \hat{n}_{-q+d/2} \rangle dq}{\int \langle \hat{n}_{q+d/2} \rangle \langle \hat{n}_{-q+d/2} \rangle dq} \]
Collapse and revival experiments with bosons in an optical lattice


Dynamical evolution of the interference pattern after jumping the optical lattice potential

Perfect correlations “hiding” in the image
Quantum noise in interference experiments with independent condensates
Interference of independent condensates


Theory: Javanainen, Yoo, PRL 76:161 (1996)
and many more
INTERFERENCE FRINGES PRODUCED BY SUPERPOSITION OF TWO INDEPENDENT MASER LIGHT BEAMS

By G. MAGYAR and Dr. L. MANDEL

Department of Physics, Imperial College of Science and Technology, London

Fig. 2. An example of fringes recorded: (a) photograph; (b) microphotometer tracing
Experiments with 2D Bose gas

Experiments with 1D Bose gas
S. Hofferberth et al. arXiv0710.1575
Interference of two independent condensates

\[ \psi(r) = \psi_1(r) + \psi_2(r) \]
\[ \rho_{\text{int}}(r) = \psi_1^\dagger(r) \psi_2(r) + \text{c.c.} \]
\[ \psi_1(r) = e^{i\phi_1 + ik_1 r} \quad k_1 = \frac{m v_1}{\hbar} = \frac{mr}{\hbar t} \]
\[ \psi_2(r) = e^{i\phi_2 + ik_2 r} \quad k_2 = \frac{m v_2}{\hbar} = \frac{m(r + d)}{\hbar t} \]
\[ \rho_{\text{int}}(r) = e^{i(k_2 - k_1) r} e^{i(\phi_2 - \phi_1)} + \text{c.c.} \]
\[ \rho_{\text{int}}(r) = e^{i \frac{m d r}{\hbar t}} e^{i(\phi_2 - \phi_1)} + \text{c.c.} \]

Clouds 1 and 2 do not have a well defined phase difference. However each individual measurement shows an interference pattern

\[ \langle \rho_{\text{int}}(r) \rangle = 0 \]
\[ \langle \rho_{\text{int}}(r) \rho_{\text{int}}(r') \rangle = e^{i \frac{m d}{\hbar t} (r - r')} + \text{c.c.} \]
Interference of fluctuating condensates

Polkovnikov, Altman, Demler, PNAS 103:6125(2006)

Amplitude of interference fringes, $A_{fr}$

$$|A_{fr}| e^{i\Delta \phi} = \int_{0}^{L} dx \ e^{i(\phi_1(x)-\phi_2(x))}$$

For independent condensates $A_{fr}$ is finite but $\Delta \phi$ is random

$$\langle |A_{fr}|^2 \rangle = \int_{0}^{L} dx_1 \int_{0}^{L} dx_2 \ \langle e^{i(\phi_1(x_1)-\phi_2(x_2))} \ e^{-i(\phi_1(x_2)-\phi_2(x_2))} \rangle$$

$$\langle |A_{fr}|^2 \rangle \approx L \int_{0}^{L} dx \ \langle e^{i(\phi_1(x)-\phi_1(0))} \rangle \langle e^{-i(\phi_2(x)-\phi_2(0))} \rangle$$

For identical condensates

$$\langle |A_{fr}|^2 \rangle = L \int_{0}^{L} dx \ (G(x))^2$$

Instantaneous correlation function

$$G(x) = \langle e^{i\phi(x)} e^{-\phi(0)} \rangle$$
Fluctuations in 1d BEC

For a review see Shlyapnikov et al., J. Phys. IV France 116, 3-44 (2004)

Thermal fluctuations

Thermally energy of the superflow velocity

\[ V_s = \nabla \phi(x) \]

\[ \xi_T = \sqrt{\frac{\hbar^2 m}{T}} \]

Quantum fluctuations

\[ \langle e^{i\phi(x_1)} e^{i\phi(x_2)} \rangle \sim \left( \frac{\xi_h}{|x_1 - x_2|} \right)^{1/2K} \]

\[ K = \sqrt{\frac{n}{g m}} \]
Interference between Luttinger liquids

Luttinger liquid at $T=0$

For non-interacting bosons

$$K = \infty \quad \text{and} \quad A_{fr} \sim L$$

For impenetrable bosons

$$K = 1 \quad \text{and} \quad A_{fr} \sim \sqrt{L}$$

Finite temperature

Experiments: Hofferberth, Schumm, Schmiedmayer

$$n_{1d} = 60 \mu m^{-1}$$

$$K = 47$$

$$T_{fit} = 84 \pm 22 \text{ nK}$$
Distribution function of fringe amplitudes for interference of fluctuating condensates

Gritsev, Altman, Demler, Polkovnikov, Nature Physics 2006
Imambekov, Gritsev, Demler, cond-mat/0612011

\[ A_{fr} \] is a quantum operator. The measured value of \( |A_{fr}| \) will fluctuate from shot to shot.

\[
\langle |A_{fr}|^{2n} \rangle = \int_0^L dz_1 \ldots dz_n \langle e^{i\phi(z_1)} \ldots e^{i\phi(z_n)} e^{-i\phi(z'_1)} \ldots e^{-i\phi(z'_n)} \rangle^2
\]

Higher moments reflect higher order correlation functions

We need the full distribution function of \( |A_{fr}| \)
Distribution function of interference fringe contrast

Experiments: Hofferberth et al., arXiv0710.1575
Theory: Imambekov et al., cond-mat/0612011

Quantum fluctuations dominate: asymmetric Gumbel distribution (low temp. T or short length L)

Thermal fluctuations dominate: broad Poissonian distribution (high temp. T or long length L)

Intermediate regime: double peak structure

Comparison of theory and experiments: no free parameters
Higher order correlation functions can be obtained
Distribution function of fringe amplitudes for interference of fluctuating condensates

Gritsev, Altman, Demler, Polkovnikov, Nature Physics 2006
Imambekov, Gritsev, Demler, cond-mat/0612011

$A_{fr}$ is a quantum operator. The measured value of $|A_{fr}|$ will fluctuate from shot to shot.

$$\langle |A_{fr}|^{2n} \rangle =$$

$$\int_{0}^{L} dz_1 \ldots dz_n \left| \langle e^{i\phi(z_1)} \ldots e^{i\phi(z_n)} e^{-i\phi(z'_1)} \ldots e^{-i\phi(z'_n)} \rangle \right|^2$$

Higher moments reflect higher order correlation functions

We need the full distribution function of $|A_{fr}|$
Calculating distribution function of interference fringe amplitudes

Method I: mapping to quantum impurity problem

Change to periodic boundary conditions
(long condensates)

\[ \langle |A_{fr}|^{2n} \rangle = \langle |A_{fr}|^2 \rangle^n \times Z_{2n} \]

\[ Z_{2n} = \prod_{i<j} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{du_i}{2\pi} \frac{dv_j}{2\pi} \left| \frac{\prod_{i<j} 2 \sin\left( \frac{u_i - u_j}{2} \right) \prod_{i<j} 2 \sin\left( \frac{v_i - v_j}{2} \right)}{\prod_{i<j} 2 \sin\left( \frac{u_i - v_j}{2} \right)} \right|^{1/K} \]

Explicit expressions for \( Z_{2n} \) are available but cumbersome
Impurity in a Luttinger liquid

\[ S = \frac{\pi K}{2} \int dx \, d\tau \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + 2g \int d\tau \cos \phi (x = 0, \tau) \]

Expansion of the partition function in powers of \( g \)

\[ Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(2n)!} \int d\tau_1 \ldots d\tau_n (e^{i\phi} + e^{-i\phi})_{\tau_1} \ldots (e^{i\phi} + e^{-i\phi})_{\tau_{2n}} \]

\[ Z_{\text{imp}} = \sum \frac{g^{2n}}{(n!)^2} Z_{2n} \]

\[ Z_{2n} = \prod_{i<j} \int_0^{2\pi} \int_0^{2\pi} \frac{du_i}{2\pi} \frac{dv_j}{2\pi} \left| \frac{\prod_{i<j} 2 \sin \left( \frac{u_i - u_j}{2} \right) \prod_{i<j} 2 \sin \left( \frac{v_i - v_j}{2} \right)}{\prod_{ij} 2 \sin \left( \frac{u_i - v_j}{2} \right)} \right|^{1/K} \]

Partition function of the impurity contains correlation functions taken at the same point and at different times. Moments of interference experiments come from correlations functions taken at the same time but in different points. Euclidean invariance ensures that the two are the same.
Relation between quantum impurity problem and interference of fluctuating condensates

Normalized amplitude of interference fringes

\[ a^2 = \frac{|A_{fr}|^2}{\langle |A_{fr}|^2 \rangle} \]

Distribution function of fringe amplitudes

\[ W(K, a^2) \]

Relation to the impurity partition function

\[ Z_{imp}(K, g) = \int_0^\infty da^2 W(K, a^2) I_0(2ga) \]

Distribution function can be reconstructed from

\[ W(K, a^2) = 2 \int_0^\infty g \, dg \, Z_{imp}(K, ig) J_0(2ga^2) \]
Bethe ansatz solution for a quantum impurity

$Z_{\text{imp}}(K, g)$ can be obtained from the Bethe ansatz following
Making analytic continuation is possible but cumbersome

Interference amplitude and spectral determinant

$Z_{\text{imp}}(K, ig)$ is related to a Schroedinger equation


$$-\frac{d^2 \Psi}{dx^2} + \left( x^4K^{-2} + \frac{3}{4x^2} \right) \Psi = E \Psi$$

Spectral determinant

$$D(E) = \prod_{n=1}^{\infty} \left( 1 - \frac{E}{E_n} \right)$$

$$Z_{\text{imp}}(K, ig) = D \left( \frac{g^2}{\pi^2} (4K)^{2-K^{-1}} \left[ \Gamma\left(1 - \frac{1}{2K}\right) \right]^2 \sin^2\left(\frac{\pi}{2K}\right) \right)$$
Interference of 1d condensates at $T=0$.
Distribution function of the fringe contrast

$$a^2 = \frac{|A_{fr}|^2}{\langle |A_{fr}|^2 \rangle}$$

Narrow distribution for $K \to \infty$
Approaches Gumbel distribution.

Width $\sigma = \frac{\pi}{6K}$

Wide Poissonian distribution for $K \to 1$
When $K>1$, $Z_{\text{imp}}(K, ig)$ is related to $Q$ operators of CFT with $c<0$. This includes 2D quantum gravity, non-intersecting loop model on 2D lattice, growth of random fractal stochastic interface, high energy limit of multicolor QCD, ...