Part II

New challenges in quantum many-body theory: non-equilibrium coherent dynamics
Non-equilibrium dynamics of many-body systems of ultracold atoms

1. Dynamical instability of strongly interacting bosons in optical lattices
2. Adiabaticity of creating many-body fermionic states in optical lattices
3. Dynamical instability of the spiral state of F=1 ferromagnetic condensate
4. Dynamics of coherently split condensates
5. Many-body decoherence and Ramsey interferometry
6. Quantum spin dynamics of cold atoms in an optical lattice
Dynamical Instability of the Spiral State of F=1 Ferromagnetic Condensate

Ref:
R. Cherng et al, arXiv:0710.2499
Ferromagnetic spin textures created by D. Stamper-Kurn et al.

- generate helical spin pattern (uniform spin current) using inhomogeneous field gradient

\[ \frac{dB_z}{dz} \Rightarrow \text{Evolve} \]

\( w/o \) gradient

\( \text{with gradient} \)
Dissolving spin textures

initial texture = uniform

initial texture = wound up
F=1 condensates

Spinor order parameter
Vector representation

Ferromagnetic State

\[
S_x = \cos \phi \quad S_y = \sin \phi
\]

\[
\vec{\Psi} = \frac{\Psi_0}{\sqrt{2}} \times \begin{pmatrix}
    i \sin \phi \\
    i \cos \phi \\
    1
\end{pmatrix}
\]

Polar (nematic) state

\[
\vec{\Psi} = \Psi_0 \times \begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix}
\]

\[
\mathcal{H} = \frac{1}{2m} \nabla \Psi_{\alpha}^\dagger \nabla \Psi_{\alpha} + q F_z^2 + \frac{g_0}{2} \Psi_{\alpha}^\dagger \Psi_{\beta}^\dagger \Psi_{\beta} \Psi_{\alpha} + \frac{g_s}{2} \Psi_{\alpha}^\dagger \Psi_{\alpha}^\dagger \Psi_{\beta} \Psi_{\beta}
\]

Ferromagnetic state realized for \( g_s > 0 \)
Spiral Ferromagnetic State of F=1 condensate

Gross-Pitaevski equation

\[ i \frac{\partial \Psi_\alpha}{\partial t} = -\frac{\nabla^2}{2m} \Psi_\alpha + g_0 \Psi_\beta^\dagger \Psi_\beta \Psi_\alpha + g_s \Psi_\alpha^\dagger \Psi_\beta \Psi_\beta \]

Mean-field spiral state

The nature of the mean-field state depends on the system preparation.

Sudden twisting

Adiabatic limit: \( \theta \) determined from the condition of the stationary state.

\[ \Psi(x, t) = \Psi_0 \left( \begin{array}{c}
  i \cos qx \frac{1}{\sqrt{2}} e^{-i\omega_1 t} \\
  i \sin qx \frac{1}{\sqrt{2}} e^{-i\omega_1 t} \\
  \frac{1}{\sqrt{2}} e^{-i\omega_2 t}
\end{array} \right) \]

\[ \tilde{\Psi}(x) = \Psi_0 \left( \begin{array}{c}
  i \cos qx \cos \theta \\
  i \sin qx \cos \theta \\
  \sin \theta
\end{array} \right) e^{-i\mu t} \]

Instabilities can be obtained from the analysis of collective modes
Collective modes
Instabilities of the spiral state

Adiabatic limit

Sudden limit
Mean-field energy

Inflection point suggests instability

Negative value of $\partial^2 E_{MF}/\partial q^2$ shows that the system can lower its energy by making a non-uniform spiral winding.
Instabilities of the spiral state

Beyond mean-field: thermal and quantum phase slips?
Dynamics of coherently split condensates. Interference experiments

Refs:
Bistrizer, Altman, PNAS 104:9955 (2007)
Interference of one dimensional condensates

Studying dynamics using interference experiments

Prepare a system by splitting one condensate
Take to the regime of zero tunneling
Measure time evolution of fringe amplitudes
Finite temperature phase dynamics

\[ \mathcal{H}_0 = \int dx \left[ g n_1^2(x) + \rho (\partial_x \phi_1)^2 \right] + \int dx \left[ g n_2^2(x) + \rho (\partial_x \phi_2)^2 \right] \]

Temperature leads to phase fluctuations within individual condensates

Interference experiments measure only the relative phase

\[ \phi_{av} = \frac{\phi_1 + \phi_2}{2} \]

\[ \phi = \phi_1 - \phi_2 \]
Relative phase dynamics

\[ H = \int d^d r \left[ \frac{g}{2} (\Delta n)^2 + \frac{\rho}{2} (\nabla \phi)^2 \right] \]

Hamiltonian can be diagonalized in momentum space

A collection of harmonic oscillators with \( \omega_q = \sqrt{g \rho} |q| \)

Initial state \( \phi_q = 0 \)

Need to solve dynamics of harmonic oscillators at finite \( T \)

Coherence

\[ \langle \Psi(t) | e^{i\phi} | \Psi(t) \rangle = e^{-\frac{1}{2} \sum_q \langle \phi_q^2(t) \rangle} \]
Relative phase dynamics

High energy modes, $\hbar \omega_{osc} > k_B T$, quantum dynamics
Low energy modes, $\hbar \omega_{osc} < k_B T$, classical dynamics

Combining all modes

$$t < \frac{\hbar}{k_B T}$$ Quantum dynamics

$$t > \frac{\hbar}{k_B T}$$ Classical dynamics

For studying dynamics it is important to know the initial width of the phase
Relative phase dynamics

Naive estimate

$$\delta N \sim \sqrt{N}$$

$$\frac{N}{2} \pm \delta N \quad \frac{N}{2} \mp \delta N$$

Long Phase Coherence Time and Number Squeezing of two Bose-Einstein Condensates on an Atom Chip

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Physics Department, Harvard University, Cambridge, MA 02138, USA
(Dated: August 27, 2006)
Relative phase dynamics

Separating condensates at finite rate

Instantaneous Josephson frequency
\[ \omega_J = \sqrt{UJ} \]

Adiabatic regime
\[ \dot{\omega}_J < \omega_J^2 \]

Instantaneous separation regime
\[ \dot{\omega}_J > \omega_J^2 \]

Adiabaticity breaks down when
\[ \omega_J \sim 1/\tau_s \]

Charge uncertainty at this moment
\[ U \delta N^2 \sim \omega_J \sim 1/\tau_s \]

Squeezing factor
\[ \frac{\delta N}{\sqrt{N}} \sim \sqrt{\frac{1}{NU\tau_s}} \sim \sqrt{\frac{1}{\mu\tau_s}} \]
Relative phase dynamics

Bistrizer, Altman, PNAS (2007)
Burkov, Lukin, Demler, PRL (2007)

Quantum regime

\[ \frac{h}{\mu} < t < \frac{h}{k_B T} \]

1D systems

\[ \langle e^{i\phi(t)} \rangle \sim e^{-\frac{\mu t^2}{2N\tau_s}} e^{-t/2\pi K\tau_s} \]

2D systems

\[ \langle e^{i\phi(t)} \rangle \sim e^{-\frac{\mu t^2}{2N\tau_s}} \left( \frac{t_0}{t} \right)^{1/16T_K T\tau_s} \]

Different from the earlier theoretical work based on a single mode approximation, e.g. Gardiner and Zoller, Leggett

Classical regime

\[ t > \frac{h}{k_B T} \]

1D systems

\[ \langle e^{i\phi(t)} \rangle \sim e^{-\left(\frac{t}{t_T}\right)^{2/3}} \]

2D systems

\[ \langle e^{i\phi(t)} \rangle \sim \left( \frac{t_0}{t} \right)^{\frac{T}{8T_K T}} \]

\[ t_T \sim \frac{\mu K}{T^2} \]
1d BEC: Decay of coherence


\[ \Psi(t) \propto \exp\left[-\left(\frac{t}{t_0}\right)^{2/3}\right] \]

double logarithmic plot of the coherence factor

slopes: \(0.64 \pm 0.08, 0.67 \pm 0.1, 0.64 \pm 0.06\)

get \(t_0\) from fit with fixed slope \(2/3\) and calculate \(T\) from

\[ t_0 = \frac{2.61 \pi K}{T^2} \]

\(T_5 = 110 \pm 21\text{ nK}\)
\(T_{10} = 130 \pm 25\text{ nK}\)
\(T_{15} = 170 \pm 22\text{ nK}\)
Dynamics of partially split condensates. From the Bethe ansatz solution of the quantum Sine-Gordon model to quantum dynamics

Refs:

Coupled 1d systems

\[ \mathcal{H}_0 = \int dx \left[ g n_1^2(x) + \rho (\partial_x \phi_1)^2 \right] + \int dx \left[ g n_2^2(x) + \rho (\partial_x \phi_2)^2 \right] \]

Interactions lead to phase fluctuations within individual condensates

\[ \mathcal{H}_{tun} = -J \int dx \cos(\phi_1 - \phi_2) \]

Tunneling favors aligning of the two phases

Interference experiments measure only the relative phase

\[ \phi_{av} = \frac{\phi_1 + \phi_2}{2} \quad \quad \phi = \phi_1 - \phi_2 \]
**Coupled 1d systems**

Conjugate variables

\[ \phi = \phi_1 - \phi_2 \quad \Delta n = \left( n_1 - n_2 \right)/2 \]

Relative phase

\[ \left[ \Delta n(x_1), \phi(x_2) \right] = -i \delta(x_1 - x_2) \]

Particle number imbalance

\[ \mathcal{H}[\phi] = \int dx\, d\tau \left[ \frac{1}{2K} (\Delta n)^2 + \frac{K}{2} (\partial_x \phi)^2 \right] - J \int dx\, d\tau \cos \phi \]

Small K corresponds to strong quantum fluctuations
Quantum Sine-Gordon model

Hamiltonian

$$\mathcal{H}[\phi] = \int dx \, d\tau \left[ \frac{1}{2K} (\Delta n)^2 + \frac{K}{2} (\partial_x \phi)^2 \right] - J \int dx \, d\tau \cos \phi$$

Imaginary time action

$$S[\phi] = \frac{K}{2} \int dx \, d\tau \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + J \int dx \, d\tau \cos \phi$$

Quantum Sine-Gordon model is exactly integrable

Excitations of the quantum Sine-Gordon model

\[ \phi = 2\pi \quad \text{soliton} \]

\[ \phi = 0 \quad \text{antisoliton} \]

many types of breathers
Dynamics of quantum sine-Gordon model

Hamiltonian formalism

\[ \mathcal{H}[\phi] = \int dx \, d\tau \left[ \frac{1}{2K} (\Delta n)^2 + \frac{K}{2} (\partial_x \phi)^2 \right] - J \int dx \, d\tau \, \cos \phi \]

Initial state
\[ \phi(t = 0) = 0 \]

Quantum action in space-time

\[ S[\phi] = \frac{K}{2} \int dx \, d\tau \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + J \int dx \, d\tau \, \cos \phi \]

Initial state provides a boundary condition at \( t=0 \)

Solve as a boundary sine-Gordon model
Boundary sine-Gordon model

Exact solution due to Ghoshal and Zamolodchikov (93)
Applications to quantum impurity problem: Fendley, Saleur, Zamolodchikov, Lukyanov,…

\[
S = \int_{x\tau} \left[ \frac{K}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \frac{K}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - m \cos \phi \right] - M \int_{\tau} \cos \frac{\phi(x = 0)}{2}
\]

Limit \( M \to \infty \) enforces boundary condition \( \phi(x = 0) = 0 \)

Sine-Gordon + boundary condition in space
quantum impurity problem

Boundary Sine-Gordon Model
space and time enter equivalently

Sine-Gordon + boundary condition in time
two coupled 1d BEC
Boundary sine-Gordon model

Initial state is a generalized squeezed state

\[ |\psi(t = 0)\rangle = e^{\{ \sum_\gamma g_\gamma A_\gamma^\dagger(\theta=0) + \sum_{\alpha\beta} \int_\theta K_{\alpha\beta}(\theta) A_\alpha^\dagger(-\theta) A_\beta^\dagger(\theta) \}} |\text{vac}\rangle \]

\( A_\alpha^\dagger(\theta) \) creates solitons, breathers with rapidity \( \theta \)

\( A_\gamma^\dagger(\theta = 0) \) creates even breathers only

Matrix \( K_{\alpha\beta}(\theta) \) and \( g_\gamma \) are known from the exact solution of the boundary sine-Gordon model

Time evolution

\[ A_\alpha^\dagger(\theta, t) = A_\alpha^\dagger(\theta) e^{-iE_\alpha(\theta)t} \]

Coherence

\[ \langle \psi(t) | e^{i\phi} | \psi(t) \rangle \]

Matrix elements can be computed using form factor approach
Smirnov (1992), Lukyanov (1997)
Quantum Josephson Junction

\[ H = \frac{g}{2} n^2 - J \cos \phi \]

Limit of quantum sine-Gordon model when spatial gradients are forbidden

Initial state

\[ |\psi(t = 0)\rangle = \sum_n C_{2n} |2n\rangle \]

Eigenstates of the quantum Jos. junction Hamiltonian are given by Mathieu’s functions

Time evolution

\[ |\psi(t)\rangle = \sum_n C_{2n} e^{-iE_{2n}t} |2n\rangle \]

Coherence

\[ \langle \psi(t) | e^{i\phi} | \psi(t) \rangle \]
Dynamics of quantum Josephson Junction

Power spectrum

$$P(\omega) = \left| \int e^{i\omega t} \langle \psi(t) | e^{i\phi} | \psi(t) \rangle \right|^2$$

Main peak

$$\omega = E_2 - E_0$$

“Higher harmonics”

$$\omega = E_4 - E_0, E_6 - E_0, \ldots$$

Smaller peaks

$$\omega = E_{2n+2} - E_{2n}, E_{2n+4} - E_{2n}, \ldots$$
Dynamics of quantum sine-Gordon model

\[ |\psi(t)\rangle = e^{\left\{ \sum_{\gamma} g_{\gamma} A_{\gamma}^\dagger (\theta=0) + \sum_{\alpha\beta} K_{\alpha\beta}(\theta) A_{\alpha}^\dagger (-\theta) A_{\beta}^\dagger (\theta) \right\}} |\text{vac}\rangle \]

**Coherence**

\[ \langle \psi(t) | e^{i\phi} | \psi(t) \rangle \]

**Main peak**

\[ \int_{\theta} \langle \text{vac} | e^{i\phi} | B_1(\theta) B_1(-\theta) \rangle \]

**“Higher harmonics”**

\[ \int_{\theta} \langle \text{vac} | e^{i\phi} | B_n(\theta) B_n(-\theta) \rangle \]

**Smaller peaks**

\[ \int_{\theta,\theta'} \langle B_m(\theta') B_m(-\theta') | e^{i\phi} | B_n(\theta) B_n(-\theta) \rangle \]

**Sharp peaks**

\[ \langle \text{vac} | e^{i\phi} | B_{2n}(\theta = 0) \rangle \]
Dynamics of quantum sine-Gordon model

- Main peak
- Smaller peaks
- Higher harmonics
- Sharp peaks
Many-body decoherence and Ramsey interferometry

Ref:

Ramsey interference

\[ |\Psi\rangle = e^{-iE_1 t} |\uparrow\rangle + e^{-iE_2 t} |\downarrow\rangle \]

Working with \( N \) atoms improves the precision by \( \sqrt{N} \).
Need spin squeezed states to improve frequency spectroscopy.
Squeezed spin states for spectroscopy

Motivation: improved spectroscopy, e.g. Wineland et. al. PRA 50:67 (1994)

Generation of spin squeezing using interactions. Two component BEC. Single mode approximation

\[ \mathcal{H} = \chi_s (S_{\text{tot}}^z)^2 \]

Kitagawa, Ueda, PRA 47:5138 (1993)

\[ \mathcal{H} = \int dx \left[ g_c (n_1 + n_2)^2 + g_s (n_1 - n_2)^2 + \frac{|\nabla \Psi_1|^2}{2m} + \frac{|\nabla \Psi_2|^2}{2m} \right] \]

In the single mode approximation we can neglect kinetic energy terms

\[ \mathcal{H}_{\text{SMA}} = \frac{g_s}{V} (N_1 - N_2)^2 \]
Interaction induced collapse of Ramsey fringes

Ramsey fringe visibility

\[ t_{\text{collapse}} \sim \frac{1}{\chi \sqrt{N}} \]

\[ t_{\text{revival}} \sim \frac{1}{\chi} \]

Experiments in 1d tubes:
A. Widera, I. Bloch et al.
Spin echo. Time reversal experiments

Single mode approximation

\[ \mathcal{H}_{\text{SMA}} = \frac{g_s}{V} (N_1 - N_2)^2 \]

\[ g_s = \frac{g_{11} - g_{12}}{2} \]

The Hamiltonian can be reversed by changing \( a_{12} \)

\[ a_s \to -a_s \]

\[ \mathcal{H}_{\text{SMA}} \to -\mathcal{H}_{\text{SMA}} \]

\[ e^{i \int_{T}^{2T} \mathcal{H}_{\text{SMA}}(t) dt} \times e^{i \int_{0}^{T} \mathcal{H}_{\text{SMA}}(t) dt} = 1 \]

Predicts perfect spin echo
Spin echo. Time reversal experiments

Expts: A. Widera, I. Bloch et al.

No revival?

Experiments done in array of tubes. Strong fluctuations in 1d systems. Single mode approximation does not apply. Need to analyze the full model

\[
\mathcal{H} = \int dx \left[ g_c (n_1 + n_2)^2 + g_s (n_1 - n_2)^2 + \frac{\left| \nabla \Psi_1 \right|^2}{2m} + \frac{\left| \nabla \Psi_2 \right|^2}{2m} \right]
\]
Interaction induced collapse of Ramsey fringes. Multimode analysis

Low energy effective theory: Luttinger liquid approach

Luttinger model

\[ S^+(x, t) \sim e^{i\phi_s(x,t)} \quad [S^z(x), \phi_s(x')] = -i\delta(x - x') \]

\[ \mathcal{H}_s = \int_0^L dx \left[ g_s(S^z)^2 + \frac{\rho}{2m} (\nabla \phi_s)^2 \right] \]

Changing the sign of the interaction reverses the interaction part of the Hamiltonian but not the kinetic energy

\[ \mathcal{H}_s = \sum_q \left[ g_s(t) S_q^z S_q^{z*} + \frac{\rho q^2}{m} \phi_{sq}\phi_{sq}^{*} \right] \]

\[ [S_{q'}, \phi_{sq}] = -i\delta_{qq'} \]

Time dependent harmonic oscillators can be analyzed exactly
Time-dependent harmonic oscillator

\[ \mathcal{H} = \frac{p^2}{2m(t)} + k \frac{q^2}{2} \]

Explicit quantum mechanical wavefunction can be found

\[ \psi(p, t) = \Phi\left( \frac{p}{c(t)} \right) e^{i \alpha(t) p^2 + i \gamma(t)} \]

From the solution of classical problem

\[ \ddot{c} + \omega^2(t) c = \frac{\omega_0^2}{c^3} \]

We solve this problem for each momentum component

\[ \mathcal{H}_s = \sum_q \left[ g_s(t) S_q^z S_q^{z*} + \frac{\rho q^2}{m} \phi_{sq} \phi_{sq}^* \right] \]
Interaction induced collapse of Ramsey fringes in one dimensional systems

Only $q=0$ mode shows complete spin echo. Finite $q$ modes continue decay.

The net visibility is a result of competition between $q=0$ and other modes.

Fundamental limit on Ramsey interferometry
Quantum spin dynamics of cold atoms in an optical lattice
Two component Bose mixture in optical lattice


\[ | \uparrow \rangle = | F = 1, m_F = -1 \rangle \]

\[ | \downarrow \rangle = | F = 2, m_F = -2 \rangle \]

Two component Bose Hubbard model

\[ \mathcal{H} = -t_\uparrow \sum_{\langle ij \rangle} b_{i\uparrow}^\dagger b_{j\uparrow} - t_\downarrow \sum_{\langle ij \rangle} b_{i\downarrow}^\dagger b_{j\downarrow} + U_{\uparrow\uparrow} \sum_i n_{i\uparrow} (n_{\uparrow} - 1) \]

\[ + U_{\downarrow\downarrow} \sum_i n_{i\downarrow} (n_\downarrow - 1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{\downarrow} \]
Quantum magnetism of bosons in optical lattices

\[ H = J_z \sum_{\langle ij \rangle} \sigma^z_i \sigma^z_j + J_{\perp} \sum_{\langle ij \rangle} \left( \sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j \right) \]

\[ J_z = \frac{t_{\uparrow}^2 + t_{\downarrow}^2}{2U_{\uparrow\downarrow}} - \frac{t_{\uparrow}^2}{U_{\uparrow\uparrow}} - \frac{t_{\downarrow}^2}{U_{\downarrow\downarrow}} \]

\[ J_{\perp} = - \frac{t_{\uparrow} t_{\downarrow}}{U_{\uparrow\downarrow}} \]

- Ferromagnetic
- Antiferromagnetic

\( U_{\uparrow\downarrow} \gg U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \)

\( U_{\uparrow\downarrow} \ll U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \)
Exchange Interactions in Solids

Kinetic energy dominates: \textit{antiferromagnetic} state

Coulomb energy dominates: \textit{ferromagnetic} state
Two component Bose mixture in optical lattice. Mean field theory + Quantum fluctuations

Altman et al., NJP 5:113 (2003)
Superexchange interaction in experiments with double wells

Refs:

Theory: A.M. Rey et al., arXiv:0704.1413
Experiment: S. Trotzky et al., arXiv:0712.1853
Observation of superexchange in a double well potential

Theory: A.M. Rey et al., arXiv:0704.1413

Use magnetic field gradient to prepare a state \(| \downarrow \uparrow \rangle\)

Observe oscillations between \(| \downarrow \uparrow \rangle\) and \(| \uparrow \downarrow \rangle\) states
Preparation and detection of Mott states of atoms in a double well potential
Comparison to the Hubbard model

Experiments: I. Bloch et al.

\[ \hbar \omega_{1,2} = \frac{U}{2} \left( \sqrt{\left(\frac{4J}{U}\right)^2 + 1} \right) \pm 1 \]
Beyond the basic Hubbard model

Basic Hubbard model includes only local interaction

Extended Hubbard model takes into account non-local interaction

\[
\hat{H}^{EHM} = \hat{H}^{HM} - \Delta J \sum_{\sigma \neq \sigma'} (\hat{n}_{\sigma L} + \hat{n}_{\sigma R}) \left( \hat{a}_{\sigma L}^{\dagger} \hat{a}_{\sigma' R} + \hat{a}_{\sigma' R}^{\dagger} \hat{a}_{\sigma L} \right)
\]

\[
\quad + U_{LR} \sum_{\sigma \neq \sigma'} \left( \hat{n}_{\sigma L} \hat{n}_{\sigma' R} + \hat{a}_{\sigma L}^{\dagger} \hat{a}_{\sigma' R}^{\dagger} \hat{a}_{\sigma' L} \hat{a}_{\sigma R} + \frac{1}{2} \hat{a}_{\sigma L}^{\dagger} \hat{a}_{\sigma' L} \hat{a}_{\sigma' R} \hat{a}_{\sigma R} + \frac{1}{2} \hat{a}_{\sigma R}^{\dagger} \hat{a}_{\sigma' R} \hat{a}_{\sigma' L} \hat{a}_{\sigma L} \right),
\]
Beyond the basic Hubbard model
Connecting double wells ...
Spin Dynamics of an isotropic 1d Heisenberg model

Initial state: product of triplets
Conclusions

Experiments with ultracold atoms provide a new perspective on the physics of strongly correlated many-body systems. This includes analysis of high order correlation functions, non-equilibrium dynamics, and many more.