Strongly correlated many-body systems: from electronic materials to ultracold atoms to photons

Lectures by Eugene Demler

Problems for day 2

Problem 1

Consider Hubbard model with nonlocal interactions

$$\mathcal{H} = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_i n_i + U \sum_i n_i(n_i - 1) + V_1 \sum_{\langle ij \rangle} n_i n_j + V_2 \sum_{\langle\langle ik \rangle\rangle} n_i n_k \tag{1}$$

In the limit when U is large we can keep states with occupations 0 and 1 only. This is known as the limit of hard-core bosons.

a) Show that in this limit Hamiltonian (1) can be mapped to the spin model

$$\mathcal{H} = -t \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) - h \sum_i S_i^z + V_1 \sum_{\langle ij \rangle} S_i^z S_j^z + V_2 \sum_{\langle\langle ik \rangle\rangle} S_i^z S_k^z \tag{2}$$

Discuss the relation between various spin ordered states of (2) and insulating/superfluid/supersolid states of original bosons.

b) Use Curie-Weiss type mean field approach to study the phase diagram of (2). Assume $V_2 = 0$. Keeping V_1 fixed plot the phase diagram as a function of μ and t.

c*) *more difficult part.* Extend analysis of part b) to finite V_2.

Problem 2

In this problem you will consider collapse and revival experiments with (spinless) bosonic atoms in an optical lattice (M. Greiner et al. (2002)). The system is prepared in a superfluid state. You can take this initial state to be a product of coherent states for individual wells

$$\begin{aligned} |\Psi(t=0)\rangle &= \prod_i |\alpha\rangle_i \\ |\alpha\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \end{aligned} \tag{3}$$

where $|n\rangle$ is a Fock states with n atoms in a well. At $t = 0$ the strength of the optical lattice potential is suddenly increased to a very large value, so that different wells become completely decoupled. You can take the Hamiltonian in this regime to be

$$\mathcal{H} = \frac{U}{2} \sum_i n_i(n_i - 1) \tag{4}$$

After the system evolves with the Hamiltonian (4) during time t , the TOF measurement is performed: both the periodic and parabolic confining potentials are removed, atoms expand freely, and image of the cloud is taken after long expansion.

a) Show the amplitude of interference peaks in the TOF images "collapses" after some time t then "revives", and then this cycle continues. Calculate both collapse and revival times.

b) Show that half-way between revivals the system goes through the cat state, in which $\langle b \rangle = 0$ but $\langle b^2 \rangle \neq 0$.

c)*(More difficult part) Discuss how one can detect the cat state using noise correlation analysis.

Problem 3

Consider fermionic alkaline-earth atoms with mass M and nuclear spin I trapped in an optical lattice. The internal state $|\alpha m\rangle$ of one such atom is specified by the electronic state α $(= g, e)$ and by the nuclear spin projection m, which runs over $N = 2I + 1$ nuclear Zeeman levels. In first-quantized notation, the Hamiltonian is

$$H = \sum_p H_p + \frac{1}{2} \sum_{p \neq q} H_{pq}, \tag{5}$$

where indices p, q run over the atoms. The one-body Hamiltonian is ($\hbar = 1$)

$$H_p = -\frac{1}{2M} \nabla_p^2 + \sum_\alpha |\alpha\rangle_p \langle \alpha | V_\alpha(\mathbf{r}_p), \tag{6}$$

where $V_\alpha(\mathbf{r}) = V_{0\alpha}[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)]$ is the potential seen by electronic state α, and $|\alpha\rangle_p \langle \alpha|$ projects atom p on orbital state α. Pairwise s-wave interactions are

$$H_{pq} = \delta(\mathbf{r}_p - \mathbf{r}_q) \frac{4\pi}{M} (a_{gg} |gg\rangle\langle gg| + a_{ee} |ee\rangle\langle ee| + a_{eg}^{+} |eg\rangle_{++}\langle eg| + a_{eg}^{-} |eg\rangle_{--}\langle eg|). \tag{7}$$

Here $|eg\rangle_{\pm\pm}\langle eg|$ is the projection operator on $|eg\rangle_\pm$. $|\alpha\alpha\rangle = |\alpha\rangle_p |\alpha\rangle_q$ and $|eg\rangle_\pm = (|e\rangle_p |g\rangle_q \pm |g\rangle_p |e\rangle_q)/\sqrt{2}$. $a_{\alpha\alpha}$ and $a_{eg}^{\pm}$ are the four s-wave scattering lengths.

(a) Assuming that only the lowest band is occupied, derive the Hubbard Hamiltonian

$$\begin{aligned} H' &= -\sum_{\langle j,i \rangle \alpha, m} J_\alpha (c_{i\alpha m}^\dagger c_{j\alpha m} + c_{j\alpha m}^\dagger c_{i\alpha m}) + \sum_{j,\alpha} \frac{U_{\alpha\alpha}}{2} n_{j\alpha}(n_{j\alpha} - 1) \\ &+ V \sum_j n_{je} n_{jg} + V_{ex} \sum_{j,m,m'} c_{jgm}^\dagger c_{jem'}^\dagger c_{jgm'} c_{jem} \end{aligned} \tag{8}$$

and express J_α, $U_{\alpha\alpha}$, V, and V_{ex} in terms of M, the four scattering lengths, $V_g(\mathbf{r})$, $V_e(\mathbf{r})$, and the Wannier functions $w_\alpha(\mathbf{r}-\mathbf{r_j})$, where $\mathbf{r_j}$ is the center of site j. Here $c^\dagger_{j\alpha m}$ creates an atom in internal state $|\alpha m\rangle$ at site j, $n_{j\alpha m} = c^\dagger_{j\alpha m} c_{j\alpha m}$, and $n_{j\alpha} = \sum_m n_{j\alpha m}$. The sum $\langle j, i\rangle$ is over pairs of nearest neighbor sites i, j. Constant terms, proportional to $\sum_j n_{j\alpha}$, are omitted in Eq. (8).

(b) Define SU(2) orbital algebra via

$$T^\mu = \frac{1}{2} \sum_{jm\alpha\beta} c^\dagger_{j\alpha m} \sigma^\mu_{\alpha\beta} c_{j\beta m}, \tag{9}$$

where σ^μ ($\mu = x, y, z$) are Pauli matrices in the $\{e, g\}$ basis. Verify that $[T^z, H'] = 0$. This U(1) symmetry follows from the elasticity of collisions as far as the electronic state is concerned.

(c) Define nuclear-spin permutation operators

$$S^m_n = \sum_{j,\alpha} c^\dagger_{j\alpha n} c_{j\alpha m}. \tag{10}$$

Verify that $[S^m_n, H'] = 0$ for all n, m. This SU(N) symmetry follows from the independence of scattering lengths and of the trapping potential from the nuclear spin.

(d) Derive the conditions on J_g, J_e, U_{gg}, U_{ee}, V, and V_{ex}, under which the U(1) orbital symmetry is enhanced up to a full SU(2) symmetry: $[T^z, H'] = [T^y, H'] = [T^z, H'] = 0$.