Strongly correlated many-body systems: from electronic materials to ultracold atoms to photons

Lectures by Eugene Demler

Problems for day 3

Problem 1

A common model for discussing the high T_c cuprates is the t-J model with the Hamiltonian

$$\mathcal{H} = -t \sum_{<ij>} P_s \cdot C_{i\sigma}^{+} C_{j\sigma} \cdot P_s + J \sum_{<ij>} \vec{S}_i \vec{S}_j \; ; \; J > 0.$$

Here $< ij >$ denotes the nearest neighbors and P_s projects out the states with two electrons on one site (so only states with 0 or 1 electrons per site exist in the Hilbert space of the t-J model). In this problem you will study the t-J model on a 4 site plaquette.

a) When there is exactly 4 electrons per 4 sites, the t-J model reduces to the AF Heisenberg model. Show that the ground state in this case is

$$|\psi_4> = \frac{1}{2}(S_{12}^{+} S_{34}^{+} - S_{14}^{+} S_{23}^{+}) \, |vacuum>$$

where

$$\frac{S_{ij}^{+} = (C_{i\uparrow}^{+} C_{j\downarrow}^{+} - C_{i\downarrow}^{+} C_{j\uparrow}^{+})}{\sqrt{2}}$$

Hint: Write the Heisenberg Hamiltonian as

$$J(\vec{S}_1 \vec{S}_2 + \vec{S}_2 \vec{S}_3 + \vec{S}_3 \vec{S}_4 + \vec{S}_4 \vec{S}_1) = \frac{J}{2}(\vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4)^2 - J(\vec{S}_1 \vec{S}_3 + \vec{S}_2 \vec{S}_4),$$

and construct a state for which $\vec{S}_{tot} = 0$, but $\vec{S}_1 \cdot \vec{S}_3$ and $\vec{S}_2 \cdot \vec{S}_4$ are maximized.

b) Find the ground state for two electrons $|\psi_2)$.

c) Take the definition of the $d_{x^2-y^2}$ Cooper pair creation operator, $\tilde{\Delta}_{d_{x^2-y^2}}$, as in the notes:

$$\tilde{\Delta}_{d_{x^2-y^2}}^{\dagger} = S_{12}^{\dagger} + S_{43}^{\dagger} - S_{14}^{\dagger} - S_{23}^{\dagger}$$

Compute $< \ \psi_4 \mid \tilde{\Delta}^+_{dx^2-y^2} \mid \psi_2 \ >$

d) Define the "extended s-wave" Cooper pair creation operator as

$$\tilde{\Delta}^+_{s^*} \ = \ (S^+_{12} \ + \ S^+_{43} \ + \ S^+_{14} \ + \ S^+_{23})$$

Compute $< \ \psi_4 \mid \tilde{\Delta}^+_{s^*} \mid \psi_2 \ >$

Problem 2

The Self Consistent Harmonic Approximation (SCHA) is an easy way to find the universal jump in superfluid stiffness associated with the BKT transition. The SCHA is a variational method in which we try to find the quadratic trial action that is closest to the true action in the sense of free energy. If we could diagonalize the original action, we could easily find the corresponding free energy $F = -\frac{1}{\beta}\log \mathrm{Tr} e^{-\beta S}$, where T is the temperature. However, since this is rather hard in general, we try to find an easy to compute upper bound on it.

Step 1: Show that for any trial action, the following expression is an upper bound on the true free energy

$$F_{\text{tr}} + \langle S - S_{\text{tr}} \rangle_{\text{tr}}, \tag{1}$$

where $F_{\text{tr}} = -\frac{1}{\beta}\log \mathrm{Tr} e^{-\beta S_{\text{tr}}}$ and $\langle O \rangle_{\text{tr}} = \left(\mathrm{Tr} e^{-\beta S_{\text{tr}}} O\right)\left(\mathrm{Tr} e^{-\beta S_{\text{tr}}}\right)$. Note that for any set of real quantities f,

$$\langle e^f \rangle \geq e^{\langle f \rangle}. \tag{2}$$

Step 2: We start with the lattice model for the two dimensional superfluid

$$S = -J\sum_{ij} \cos(\theta_i - \theta_j), \tag{3}$$

where J is the coupling between the lattice sites and θ_i is phase of the order parameter on the i-th site. What is the relationship (assuming no fluctuations) between J and the superfluid stiffness K. (very easy)

Step 3: Next, we introduce the trial action of the form

$$S_{\text{tr}} = -\tilde{J} \sum_{ij} (\theta_i - \theta_j)^2, \tag{4}$$

where $\tilde{J}$ is a variational parameter. Compute the correlator $\langle(\theta_i - \theta_{i+1})^2\rangle_{\text{tr}}$.

Step 4: Find an (implicit) expression for the optimal value of $\tilde{J}$ by varying the upper bound on the free energy, Eq. (1), with respect to $\tilde{J}$.

Step 5: $\tilde{J}$ is approximately the long range superfluid stiffness. Plot $\tilde{J}$ as a function of temperature, and compare your answer with the RG result.

Step 6: Make a comment about applying SCHA to cases where you expect to find no phase transition (1d) and a second order phase transition (3d).