We study periodic spin textures in dipolar spinor condensates via an low-energy effective theory describing the phase and magnetization degrees of freedom. The resulting non-linear sigma model describes the competition between kinetic energy, dipolar interactions, and a skyrmion-skyrmion interaction induced by coupling to the phase. By considering symmetry operations which combine translations, real space rotations, and spin space rotations, we classify minimal energy spin textures.

Experiments in ultracold atomic gases have provided direct and striking evidence for the theory of Bose-Einstein condensation. Typically, the combination of low temperatures and strong magnetic fields used to trap the atoms freezes out the internal hyperfine level structure. This leaves only the canonically conjugate density and phase as relevant degrees of freedom. However, recent experimental advancements have opened the possibility of exploring multicomponent condensates where the population and coherences of the hyperfine level structure play a crucial role. This includes the development of optical dipole traps used for preparation and phase-contrast imaging used for detection in multicomponent condensates.

As shown by the Berkely group, $S = 1$ $^{87}$Rb offers an ideal system for the study of multicomponent condensates. In addition to density and phase degrees of freedom familiar from single-component condensates, the magnetization naturally arises as a description of the spin degrees of freedom. A vector quantity sensitive to both the population and coherences between the three hyperfine levels, the magnetization can be directly imaged in experiments. In particular, recent results suggest the coexistence of phase coherence and a periodic array of magnetization spin domains.

The theoretical analysis of these observations is ongoing and has proceeded along a number of directions. Several papers have focused on the role of the effective dipolar interactions in spinor condensates which is modified by rapid Larmor precession induced by an applied magnetic field. This effective dipolar interaction can drive dynamical instabilities in a uniform condensate with characteristic unstable modes at wavevectors strikingly similar to that of the observed magnetization correlations. Direct numerical simulation of the full multicomponent mean-field dynamics for this system also leads to long-lived spin textures.

In this paper, we study periodic spin textures in dipolar spinor condensates from a different perspective and focus directly on the low-energy degrees of freedom. They are given by the condensate phase and magnetization orientation. Since the magnetization depends on the coherences between hyperfine levels, gradients in the magnetization can couple non-trivially to the phase. In particular, spin configuration of a topological nature such as skyrmions can induce vortex-like configurations in the phase. The resulting non-linear sigma model given by Eq. 1 for spin textures describes the competition between kinetic energy, effective dipolar interaction, and induced vortex interaction.

We consider $S = 1$ dipolar spinor condensates in a quasi-two-dimensional geometry. At energies smaller than those of the spin-independent and spin-dependent contact interactions, the local density is fixed and the magnetization is maximally polarized. Within the space of states with fully polarized magnetization, the formation of spin textures is determined by the low-energy spin-dependent interactions given by the quadratic Zeo-
man shift and dipolar interactions. The resulting effective theory is given by a non-linear sigma model describing the dynamics of the magnetization. The Lagrangian and Hamiltonian are

\[
\mathcal{L} = -\int dt d^2 x A(\hat{n}) \cdot \partial_t \hat{n} - \int dt \mathcal{H}_{KE} - \int dt \mathcal{H}_S
\]

\[
\mathcal{H}_{KE} = \frac{1}{4} \int d^2 x (\nabla \hat{n})^2 + \frac{1}{2} \int d^2 x d^2 y q(x) G(x - y) q(y)
\]

\[
\mathcal{H}_S = \int d^2 x d^2 y \hat{n}^i(x) h^{ij}(x - y) \hat{n}^j(y)
\]

where \( \hat{n} \) is a three component real unit vector giving the magnetization and \( A(\hat{n}) \) is the unit monopole vector potential, \( \mathcal{H}_{KE} \) gives kinetic energy contributions, and \( \mathcal{H}_S \) gives the low-energy spin-dependent interactions.

The first term in \( \mathcal{H}_{KE} \) is the magnetization kinetic energy while the second term comes form the superfluid kinetic energy. Non-uniform textures in \( \hat{n} \) arise in part due to phase gradients of the underlying condensate wavefunction and can couple to the superfluid velocity. This coupling fixes the curl or vorticity of the superfluid velocity field to

\[
q = \epsilon_{\mu\nu} \hat{n} \cdot \nabla_\mu \hat{n} \times \nabla_\nu \hat{n}
\]

(2)

the skyrmion density. The superfluid kinetic energy can then be written as a long-ranged vortex-vortex interaction for \( q \) by using \( -\nabla^2 G(x - y) = \delta(x - y) \), the 2D logarithmic Green’s function. The long-wavelength divergence of the skyrmion interaction arises from the gapless phase fluctuations of the superfluid.

The skyrmion density \( q \) is a topological quantity that measures how much a magnetization texture in real space covers the internal order parameter space of the sphere. For ordinary ferromagnets described by \( \mathcal{H} \) without the \( \hat{n} \) interactions, there are static solutions called skyrmions which carry net skyrmion charge. For spinor condensates, finite energy configurations must carry zero net skyrmion charge due to the long-wavelength divergence of the skyrmion interaction term. Previously, we have shown that static solutions with zero charge exist for the effective theory with skyrmion interactions. They describe neutral configurations of skyrmions and anti-skyrmions and are stabilized by the spin-dependent interactions discussed next.

The quadratic Zeeman shift and effective dipolar interaction give the spin-dependent interactions of \( \mathcal{H}_S \). The interaction tensor in momentum space is

\[
h^{ij}(k) = \frac{\hat{g}}{2} \left( \delta^{ij} + B^i B^j \right) - \frac{4\pi \hat{g}}{3} \left[ 3h kd_n - 1 \right] \left[ \delta^{ij} - 3B^i B^j \right],
\]

\[
h(k) = |B \cdot \hat{k}|^2 w(k) + |B \cdot \hat{n}|^2 |1 - w(k)|,
\]

\[
w(x) = 2x \int_0^\infty dz e^{-z(x^2 + 2z)}
\]

(3)

where \( \hat{B} \) is the direction of the external magnetic field, \( d_n \) is the thickness of the condensate normal to the plane, and \( \hat{g} = qm, \hat{g} = g_d n_{3D} m \) are the rescaled quadratic Zeeman shift and dipolar interaction strength with \( m \) the mass and \( n_{3D} \) the peak three-dimensional density. For current experiments, \( g_d n_{3D} = 0.8 \) Hz and \( q \) can be tuned on the scale of 0-20 Hz. For definiteness, we take \( q = 1.5 \) Hz and \( \hat{B} \) in the plane from here on. The form of the effective dipolar interaction is due to the combined effects of reduced dimensionality and rapid Larmor precession.

Previously, we have demonstrated that this effective dipolar interaction drives dynamical instabilities of the uniform ferromagnet. These instabilities are of two types. The first (second) is for fluctuations of \( \hat{n} \) perpendicular (parallel) to \( \hat{B} \) in spin space modulated parallel (perpendicular) to \( \hat{B} \) in real space. This suggests the instabilities favor configurations that are modulated along orthogonal directions in both real space and spin space. From Eq. 2, these are the configurations that carry non-zero skyrmion density.

Thus we see that the long-ranged effective dipolar interactions favor configurations with non-zero skyrmion density while the long-ranged skyrmion interactions enforce zero net skyrmion charge. In order to study minimal energy configurations of the theory, we must allow configurations to carry local skyrmion density but also ensure that the net skyrmion charge is zero.

To do this, we classify periodic crystalline configura-
tions via a symmetry analysis. Crystallographic groups give the symmetry groups of crystals and contain symmetry operations describing real space translations, rotations, and reflections. In particular, a discrete set of translations relates any point \( x \) within the crystal to a point \( x_T \) within the unit cell. Furthermore, point group rotations and reflections relate \( x_T \) within the unit cell to \( x_{PG} \) within a smaller fundamental region inside the unit cell. By allowing for combinations of real space operations with spin space rotations and reflections, we study symmetry groups that relate the magnetization \( \hat{n}(x) \) at a point \( x \) to \( \hat{n}(x_T) \) with \( x_T \) in the unit cell and then \( \hat{n}(x_{PG}) \) with \( x_{PG} \) in the fundamental region. With appropriate choice of the symmetry group, even though the skyrmion density can be non-zero within the fundamental region, neighboring regions are related via symmetry operations which ensure an overall neutral skyrmion charge. A depiction of the fundamental region is shown schematically in Fig. 2.

At a formal level, we consider symmetry operations specified by the triple \((R, t, O)\) where \( R \) is a \( \times 2 \) real orthogonal matrix describing real space rotations and reflections, \( t \) is a real two-component vector describing real space translations, and \( O \) is a \( \times 3 \) real orthogonal matrix describing spin space rotations and reflections. For \((R, t, O)\) to be a symmetry operation, the magnetization \( \hat{n} \) and skyrmion density \( q \) satisfy

\[
\hat{n}^i(x_\mu) = O^{ij}\hat{n}^j(R_{\mu\nu}x_\nu + t_\mu),
\]

\[
q(x_\mu) = \det(O)\det(R)q(R_{\mu\nu}x_\nu + t_\mu)
\]

where from here on greek (roman) indices denote coordinates in real (spin) space. The real space symmetry operations are shown schematically in Fig. 2. In addition to translations, there are reflections about mirror lines shown in purple, glide reflections along glide reflection lines shown in green, and rotations about rotation centers shown in white.

In momentum space, the symmetry operation is expressed as

\[
\hat{n}^i(k_\mu) = \exp(ik_\mu R^{-1}_{\mu\nu} t_\nu)O^{ij}\hat{n}^j(R_{\mu\nu}k_\nu),
\]

\[
q(k_\mu) = \exp(ik_\mu R^{-1}_{\mu\nu} t_\nu)\det(O)\det(R)q(R_{\mu\nu}k_\nu)
\]

The \( k = 0 \) component of \( q \) (\( \hat{n} \)) gives the net skyrmion charge (net magnetization). For symmetry groups containing an operation with \( \det(O)\det(R) \neq 1 \), then the net skyrmion charge \( q(k = 0) \) vanishes identically. Similarly, the components of the uniform magnetization \( \hat{n}(k = 0) \) that are not left fixed by \( O^j \) also vanish identically.

Having discussed the symmetry operations, we now turn to symmetry groups consisting of collections of symmetry operations. In the absence of a condensate, the system has full real space translational and rotational as well as full spin space rotational symmetry. The corresponding symmetry group is \( E_2 \otimes O(3) \) the direct product of \( E_2 \) the 2D Euclidean group in real space and \( O(3) \) the 3D orthogonal group in spin space. The symmetry group with the condensate is a subgroup of \( E_2 \otimes O(3) \). For a periodic crystal, the real-space symmetry operations are given by a choice of crystallographic group \( C \). This is a subgroup of \( E_2 \) containing only a discrete lattice of translations and compatible rotations and reflections.

In order to determine the compatible spin-space operations, we note that experimentally, it is observed that the magnetization is not confined to one plane or one axis. This implies there are no global spin space symmetry operations of the form \((1, 0, O)\) that do not also act on real space where \( 1 \) is the identity matrix and \( 0 \) is the zero vector. In this case, the subgroups of \( E_2 \otimes O(3) \) are given by \((R, t, \phi(R, t))\) where \( R, t \) are the real space symmetry operations of \( C \) the crystallographic group and \( \phi(R, t) \) is a real orthogonal representation of \( C \) in \( O(3) \).

We can thus classify the allowed symmetry groups through the 2D crystallographic groups and their real three-component orthogonal representations. Equivalently, this can be achieved by studying the complex two-component unitary representations and anti-unitary corepresentations. This is because for \( U \) a \( \times 2 \) unitary matrix or \( U\Theta \) a \( \times 2 \) anti-unitary matrix with \( \Theta \) the complex conjugation operator, there is a homomorphism...
FIG. 4: Two minimal energy periodic spin configurations for (p2gg, π/a, E₁) (left and right) for the real space unit cell (top) and in momentum space (bottom).

to a corresponding $3 \times 3$ real orthogonal matrix

$$R_U^{ij} = \frac{1}{2} \text{Tr} \left[ \sigma^i U^\dagger \sigma^j \right], \quad R_{AU}^{ij} = \frac{1}{2} \text{Tr} \left[ \sigma^i U^T (\sigma^j)^T U^\dagger \right]$$

(8)

with $\sigma$ the Pauli matrices. First consider the unitary representations. Then the allowed symmetry groups are given by the triple $(C,k,PG)$ where $C$ is a crystallographic group, $k$ is a two dimensional quasi-momentum vector specifying the representation of the translation subgroup, and $PG$ is a projective representation of the point group. Next consider the anti-unitary corepresentations. Then the allowed symmetry groups are given by the triple $(C',k',PG')$ where $C'$ is a Shubnikov or black and white crystallographic group, $k'$ is a two dimensional quasi-momentum vector specifying the corepresentation of the translation subgroup, and $PG'$ is a projective corepresentation of the point group.

In this paper, we consider crystallographic groups with a $D_2$ point group where $D_2$ is the dihedral group of order 2. This is because the external magnetic field explicitly breaks the full real space rotational symmetry down to $D_2$ through the effective dipolar interaction. The choice of unitary representation or anti-unitary corepresentation is determined by those that constrain the net skyrmion charge $q$ and net magnetization $\hat{n}$ to be zero.

After choosing a crystallographic group and a representation, we then discretize the magnetization $\hat{n}$ within the unit cell. Only the values within the fundamental region and its boundary are independent degrees of freedom since the symmetry operations relate points outside the fundamental region to those within. We then minimize the Hamiltonian in Eq. 1 to find minimal energy configurations. In order to accurately calculate the contribution of the long-ranged dipolar and skyrmion interactions, we treat both using Ewald summation techniques.

The configuration with the lowest energy is shown in Fig. 1. The symmetry group corresponds to an anti-unitary corepresentation of the $p2mg$ crystallographic group given by $(p2g, 2m'g', \pi/a, E_1)$ where the corresponding symmetry operations are shown in Fig. 2. The lattice constants are given by $a = 42\mu m$, $b = 45\mu m$. Here $B$ is along the $x$ axis in real space and the $z$ axis in spin space. Notice the modulations in $\hat{n}$ parallel to $\hat{B}$ in spin space in a direction perpendicular to $\hat{B}$ in real space with $\hat{n}$ fully polarized along $\pm z$ in stripes parallel to the $x$ axis. In between these stripes, modulations in $\hat{n}$ perpendicular to $\hat{B}$ in spin space in a direction parallel to $\hat{B}$ in real space. The skyrmion density is localized in oppositely charged regions where $\hat{n}$ changes rapidly. We show this configuration in momentum space in Fig. 3. Notice there is no net magnetization with no peak at zero wavevector. Although most of the weight is for peaks near zero wavevector, there is still significant weight at higher wavevectors. This can be understood as coming from the non-trivial structure of the magnetization within each unit cell.

The next two lowest energy configurations are shown in Fig. 4. The symmetry group corresponds to a unitary representation of the $p2gg$ crystallographic group given by $(p2g, 2mg, \pi/a, E_1)$. The lattice constants are given by $a = 48\mu m$ and $b = 53\mu m$ for the configuration on the left and $a = 31\mu m$ and $b = 38\mu m$ for the configuration on the right. Just as in the lowest energy configuration, there are modulations in $\hat{n}$ perpendicular (parallel) to $\hat{B}$ in spin space along directions parallel (perpendicular) to $\hat{B}$ in real space. However, the skyrmion density is delocalized into stripes in each of these configurations instead of localized as in the previous configuration. In momentum space $\hat{n}$ is similar to the previous one with significant weight in peaks at higher wavevectors.

In conclusion, we have analyzed periodic spin textures in spinor dipolar condensates. The low-energy effective theory describes a non-linear sigma model for the magnetization which includes competing long-ranged interactions. These include an effective dipolar interaction as well as logarithmic skyrmion-skyrmion interaction arising from the superfluid phase fluctuations. By using a systematic symmetry analysis, we are able to study configurations that have the non-zero skyrmion density favored by dipolar interaction while still satisfying the constraint of net zero skyrmion charge forced by the skyrmion interaction. We classified the allowed symmetry groups which contain combined real space and spin space operations and then found minimal energy configurations within each unit cell. These minimal energy configurations have a non-trivial structure involving modulations.
of the magnetization both parallel and perpendicular to the applied magnetic field as well as localized and stripe configurations for the skyrmion charge.