

# Strongly Correlated Systems of Cold Atoms

Detection of many-body quantum phases  
by measuring correlation functions

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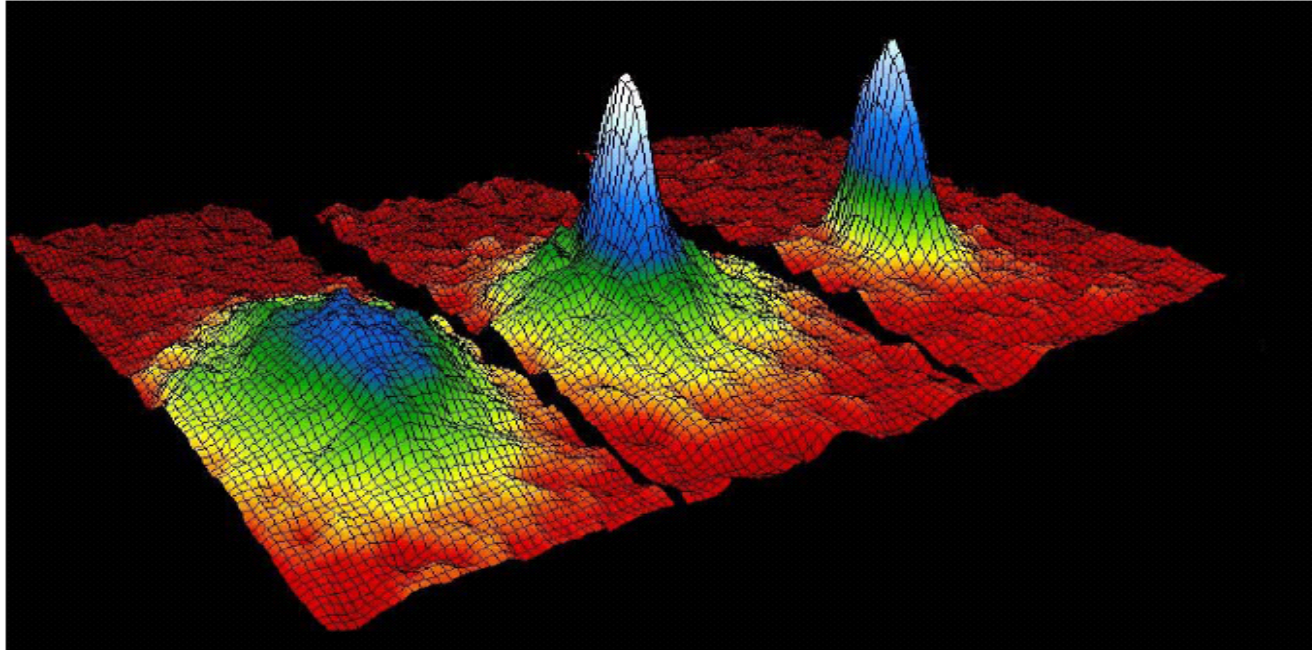
Harvard

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Thanks to: J. Schmiedmayer, M. Oberthaler, V. Vuletic,  
M. Greiner, M. Oshikawa

# Bose-Einstein condensation



Cornell et al., Science 269, 198 (1995)

$$n \sim 10^{14} \text{cm}^{-3} \quad T_{\text{BEC}} \sim 1 \mu\text{K}$$

Ultralow density condensed matter system

Interactions are weak and can be described theoretically from first principles

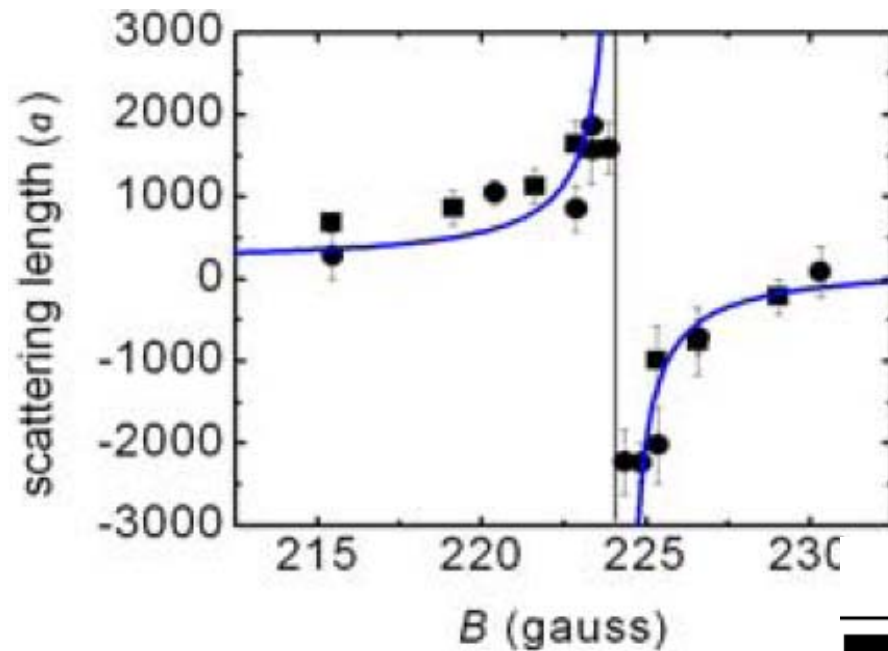
# New Era in Cold Atoms Research

## Focus on Systems with Strong Interactions

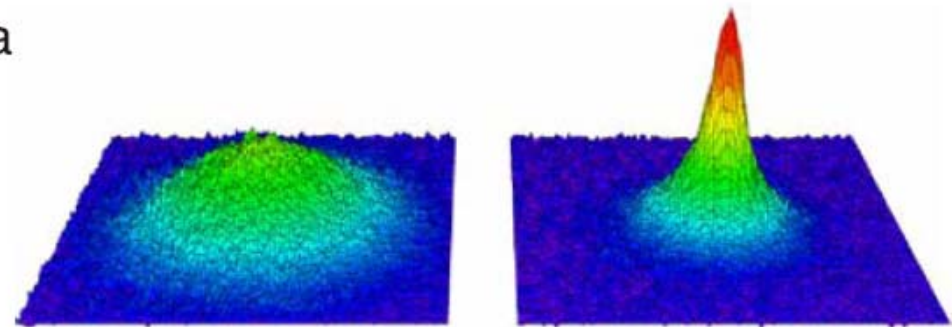
- Feshbach resonances
- Rotating systems
- Low dimensional systems
- Atoms in optical lattices
- Systems with long range dipolar interactions

# Feshbach resonance and fermionic condensates

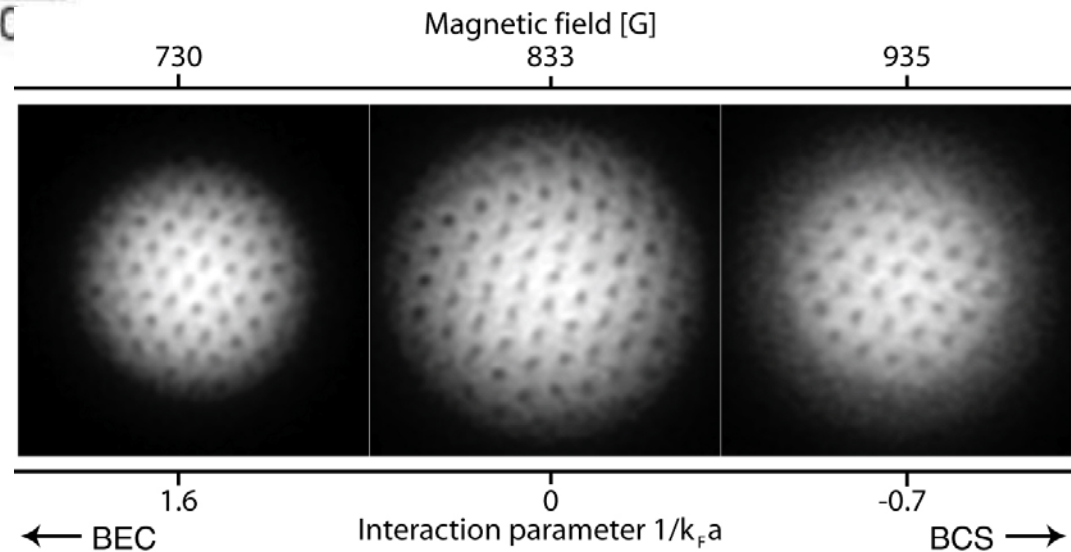
Greiner et al., Nature 426:537 (2003); Ketterle et al., PRL 91:250401 (2003)



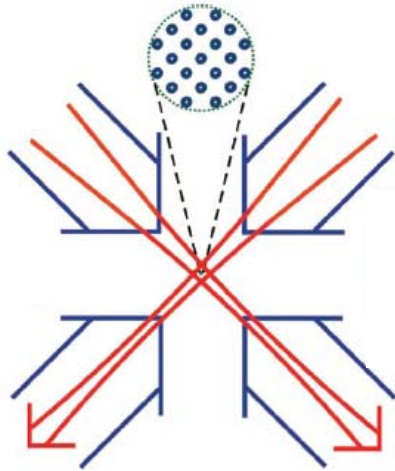
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Ketterle et al.,  
Nature 435, 1047-1051 (2005)



# One dimensional systems



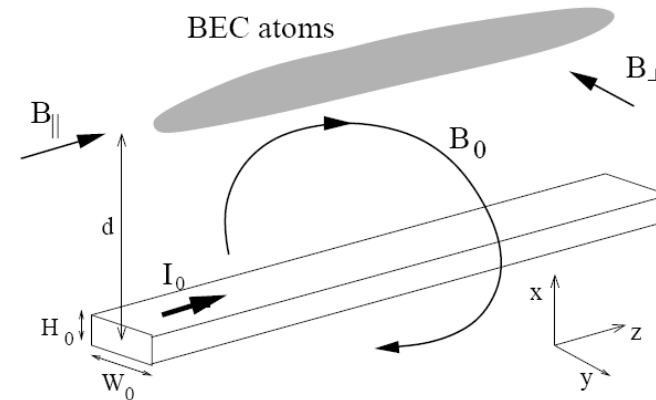
1D confinement in optical potential  
Weiss et al., Science (05);  
Bloch et al.,  
Esslinger et al.,

$$E_{\text{kin}} \sim \frac{\hbar^2}{m d^2} \sim \frac{\hbar^2 n^2}{m}$$

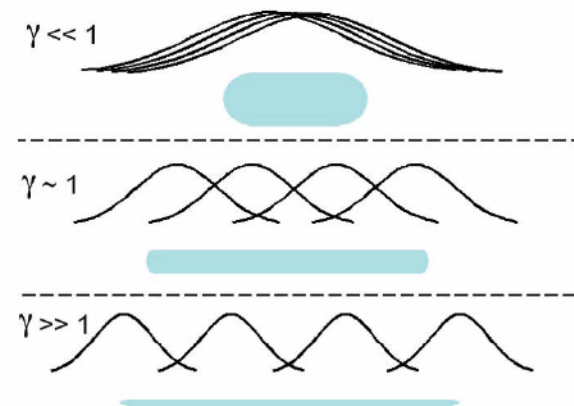
$$E_{\text{int}} \sim g n$$

$$\gamma = \frac{E_{\text{int}}}{E_{\text{kin}}} \sim \frac{g m}{\hbar^2 n}$$

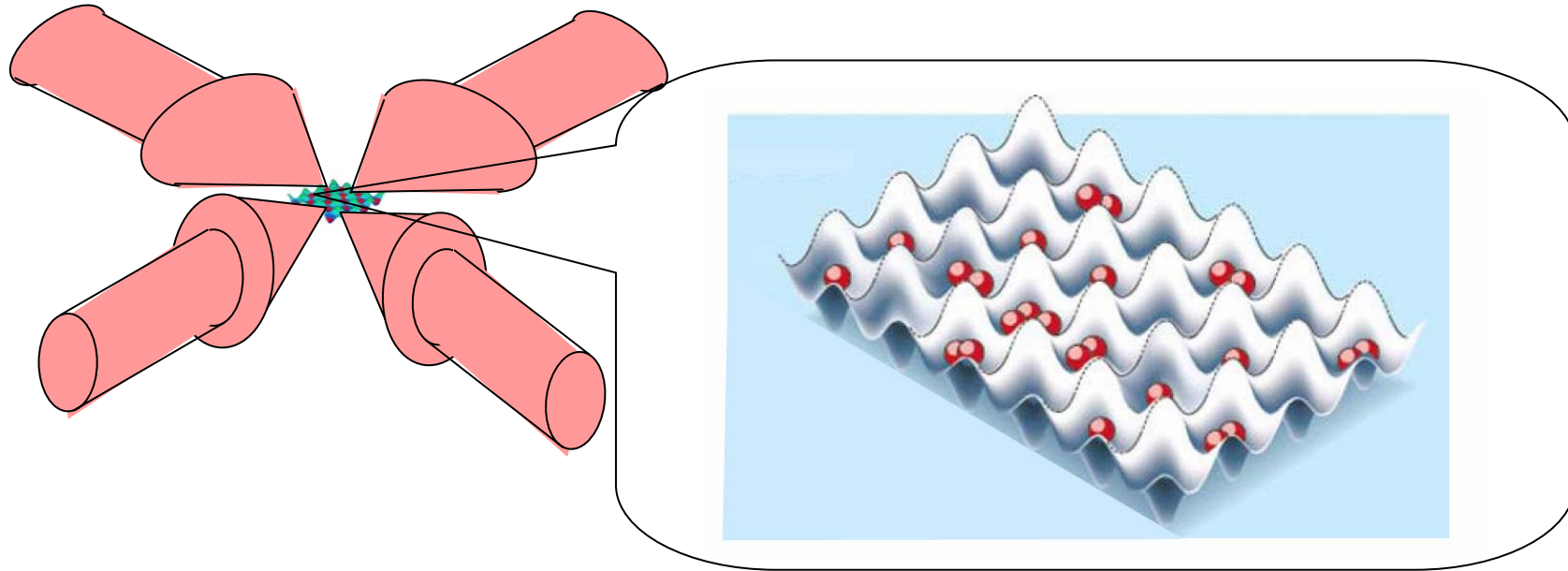
Strongly interacting  
regime can be reached  
for low densities



One dimensional systems in microtraps.  
Thywissen et al., Eur. J. Phys. D. (99);  
Hansel et al., Nature (01);  
Folman et al., Adv. At. Mol. Opt. Phys. (02)



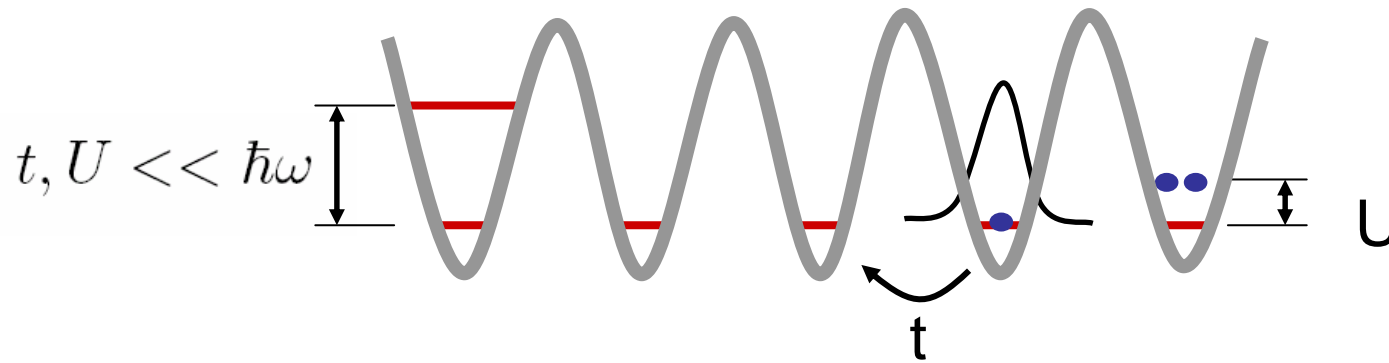
# Atoms in optical lattices



Theory: Jaksch et al. PRL (1998)

Experiment: Kasevich et al., Science (2001);  
Greiner et al., Nature (2001);  
Phillips et al., J. Physics B (2002)  
Esslinger et al., PRL (2004);  
and many more ...

# Bose Hubbard model

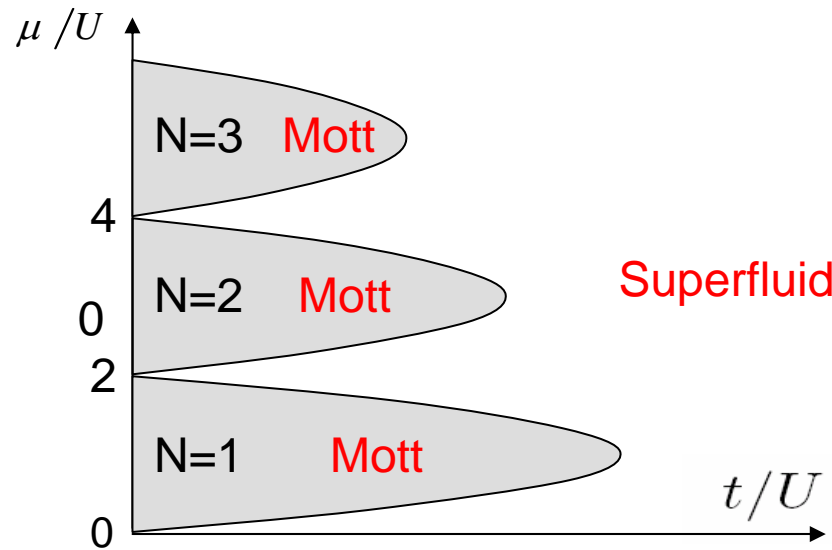


$$\mathcal{H} = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + U \sum_i n_i (n_i - 1) - \mu \sum_i n_i$$

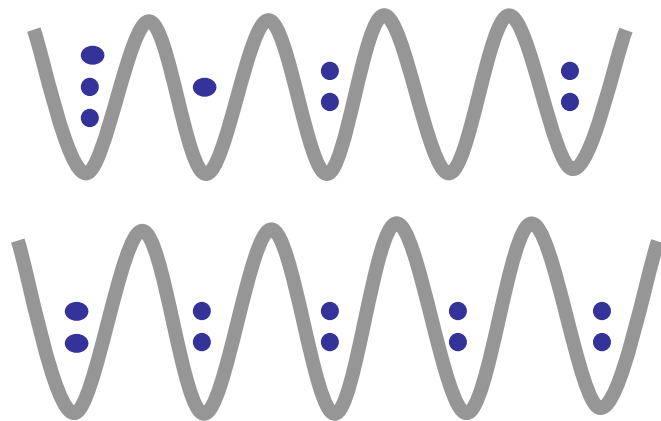
$t$  — tunneling of atoms between neighboring wells

$U$  — repulsion of atoms sitting in the same well

# Bose Hubbard model. Mean-field phase diagram



M.P.A. Fisher et al.,  
PRB40:546 (1989)



$$U \ll Nt$$

Superfluid phase  
Weak interactions

$$U \gg Nt$$

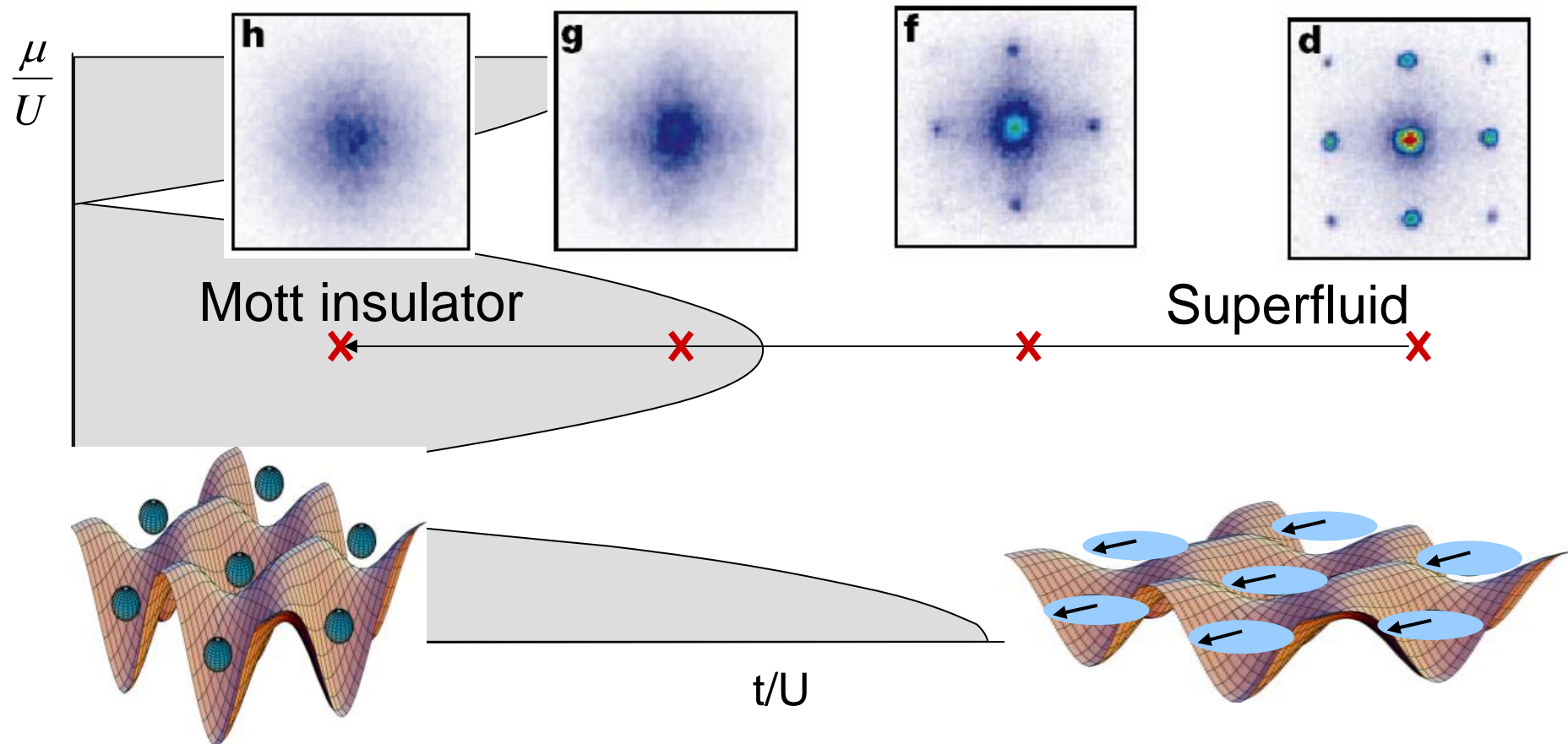
Mott insulator phase  
Strong interactions



# Superfluid to insulator transition in an optical lattice

Greiner et al., Nature 415 (2002)

See also Ketterle et al. cond-mat/0507288



# New Era in Cold Atoms Research

## Focus on Systems with Strong Interactions

### Goals

- Resolve long standing questions in condensed matter physics (e.g. origin of high temperature superconductivity)
- Resolve matter of principle questions (e.g. existence of spin liquids in two and three dimensions)
- Study new phenomena in strongly correlated systems (e.g. coherent far from equilibrium dynamics)

This talk:

Detection of many-body quantum phases  
by measuring correlation functions

# Outline

Measuring correlation functions in **interference** experiments

1. Interference of independent condensates
2. Interference of interacting 1D systems
3. Full counting statistics of interference experiments.  
Connection to quantum impurity problem
4. Interference of 2D systems

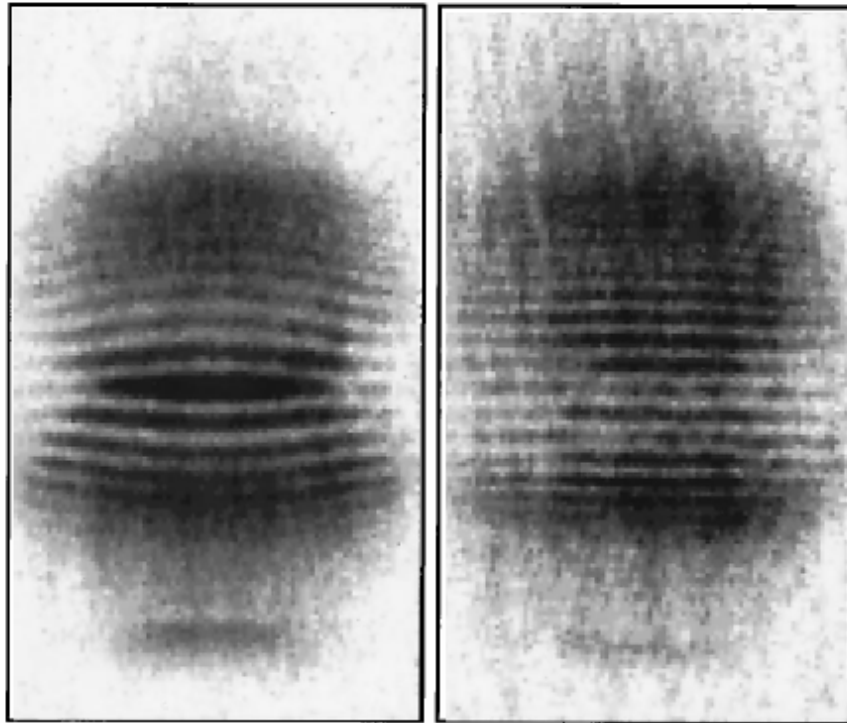
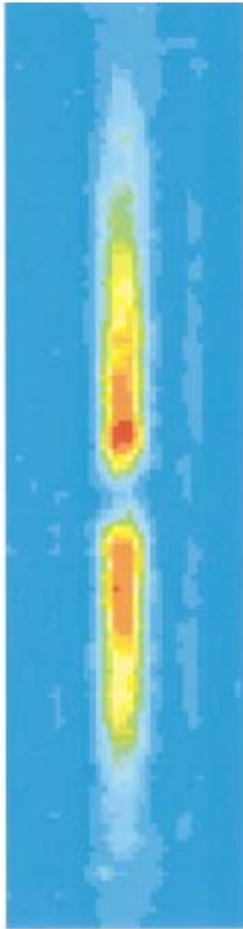
**Quantum noise** interferometry in time of flight experiments

1. Detection of magnetically ordered Mott states in optical lattices
2. Observation of fermion pairing

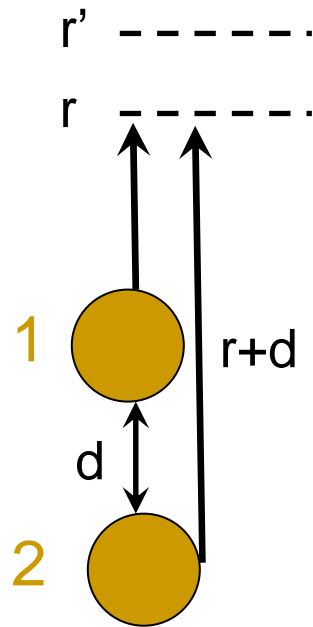
# Measuring correlation functions in interference experiments

# Interference of two independent condensates

Andrews et al., Science 275:637 (1997)



# Interference of two independent condensates



$$\rho_{\text{int}}(r) = a_1^\dagger(r) a_2(r) + \text{c.c.}$$

$$a_1(r) = e^{i\phi_1 + i k_1 r} \quad k_1 = \frac{m v_1}{\hbar} = \frac{m r}{\hbar t}$$

$$a_2(r) = e^{i\phi_2 + i k_2 r} \quad k_2 = \frac{m v_2}{\hbar} = \frac{m (r + d)}{\hbar t}$$

$$\rho_{\text{int}}(r) = e^{i(k_2 - k_1)r} e^{i(\phi_2 - \phi_1)} + \text{c.c.}$$

$$\rho_{\text{int}}(r) = e^{i \frac{m d r}{\hbar t}} e^{i(\phi_2 - \phi_1)} + \text{c.c.}$$

Clouds 1 and 2 do not have a well defined phase difference.

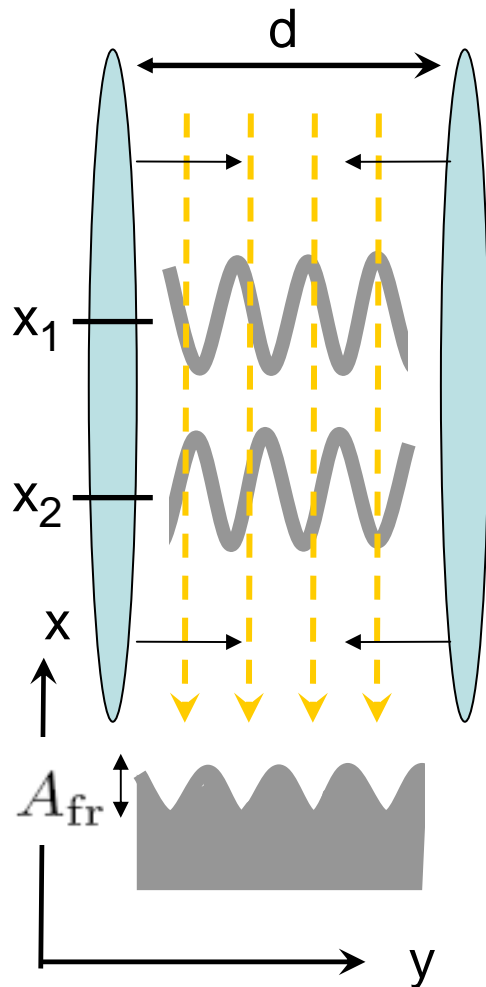
However each **individual measurement shows an interference pattern**

$$\langle \rho_{\text{int}}(r) \rangle = 0$$

$$\langle \rho_{\text{int}}(r) \rho_{\text{int}}(r') \rangle = e^{i \frac{m d}{\hbar t} (r - r')} + \text{c.c.}$$

# Interference of one dimensional condensates

Experiments: Schmiedmayer et al., Nature Physics 1 (05)



Amplitude of interference fringes,  $A_{\text{fr}}$ , contains information about phase fluctuations within individual condensates

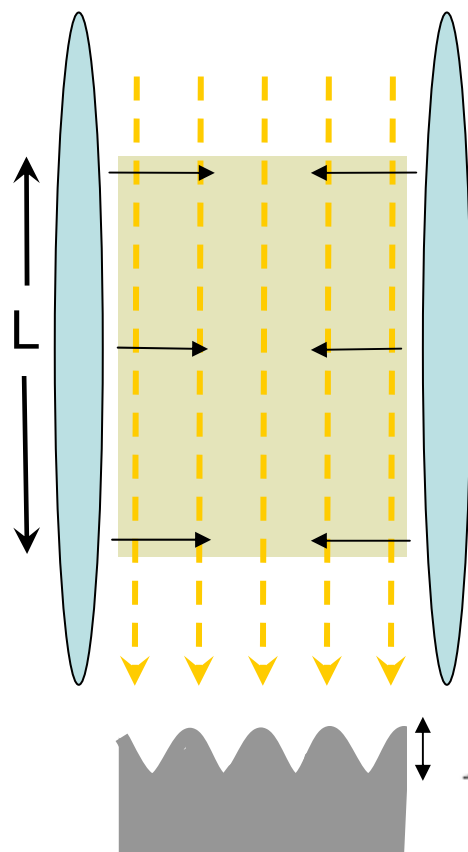
$$d\rho_{\text{int}}(x, y) = (e^{i \frac{m dy}{\hbar t}} a_1^\dagger(x) a_2(x) + \text{c.c.}) dx$$

$$\rho_{\text{int}}(y) = e^{i \frac{m dy}{\hbar t}} \int_0^L dx a_1^\dagger(x) a_2(x) + \text{c.c.}$$

$$\rho_{\text{int}}(y) = A_{\text{fr}} e^{i \Delta \phi + i \frac{m dy}{\hbar t}} + \text{c.c.}$$



# Interference amplitude and correlations



$$A_{\text{fr}} e^{i\Delta\phi} = \int_0^L dx a_1^\dagger(x) a_2(x)$$

$$\begin{aligned} \langle |A_{\text{fr}}|^2 \rangle &= \int_0^L \int_0^L dx dy \langle a_1^\dagger(x) a_2(x) a_2^\dagger(y) a_1(y) \rangle \\ &\simeq L \int_0^L dx \langle a_1(x) a_1^\dagger(0) \rangle \langle a_2(0) a_2^\dagger(x) \rangle \end{aligned}$$

For identical condensates

$$\langle |A_{\text{fr}}|^2 \rangle = L \int_0^L dx (G(x))^2$$

Instantaneous correlation function  $G(x) = \langle a(x) a^\dagger(0) \rangle$

# Interference between Luttinger liquids

Luttinger liquid at  $T=0$

$$G(x) \sim \rho \left( \frac{\xi_h}{x} \right)^{2-1/K}$$

$K$  – Luttinger parameter

$$\langle |A_{\text{fr}}|^2 \rangle \sim (\rho \xi_h)^{1/K} (L \rho)^{2-1/K}$$

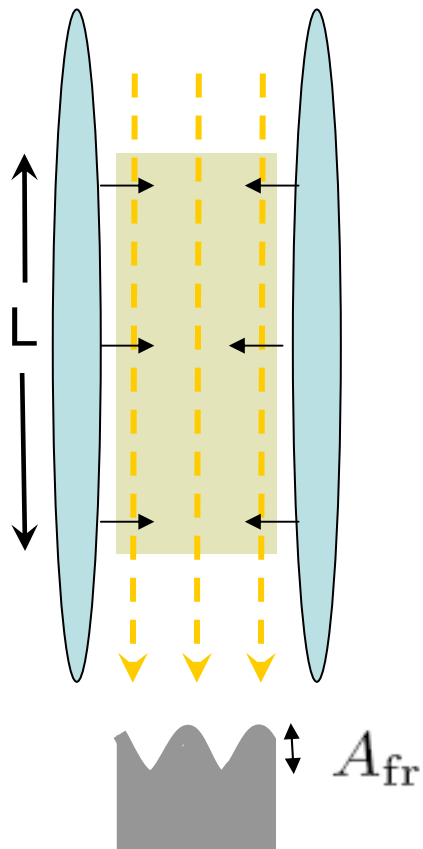
For non-interacting bosons  $K = \infty$  and  $A_{\text{fr}} \sim L$

For impenetrable bosons  $K = 1$  and  $A_{\text{fr}} \sim \sqrt{L}$

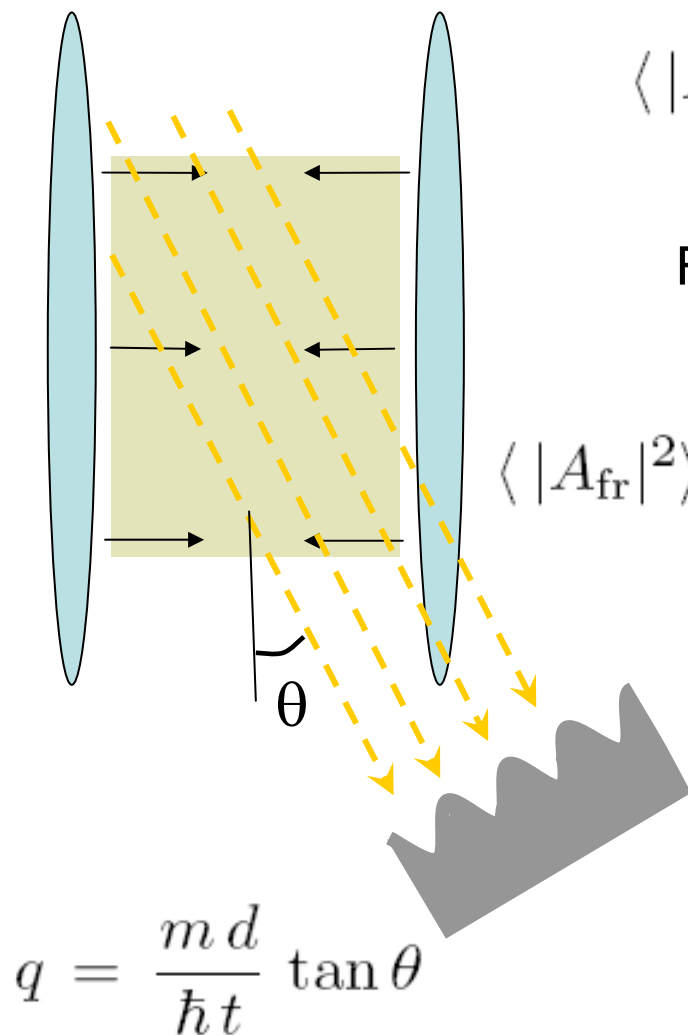
Luttinger liquid at finite temperature

$$\langle |A_{\text{fr}}|^2 \rangle \sim L \rho^2 \xi_h \left( \frac{\hbar^2}{m \xi_h^2} \frac{1}{T} \right)^{1-1/K}$$

Luttinger parameter  $K$  may be extracted from the  $L$  or  $T$  dependence of  $A_{\text{fr}}$



# Rotated probe beam experiment



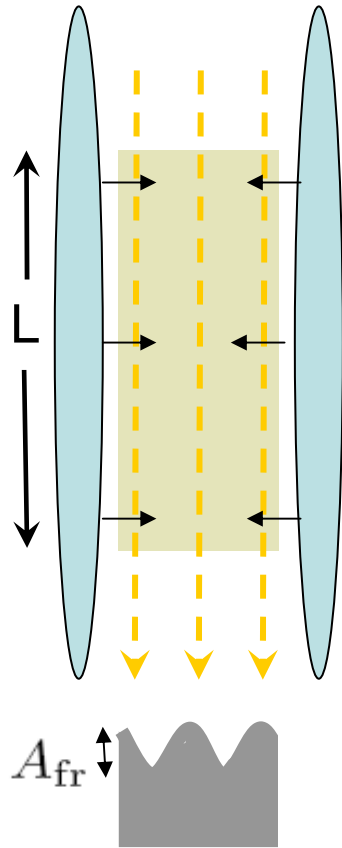
$$\langle |A_{\text{fr}}|^2 \rangle = L \int_0^L dx \cos(qx) (G(x))^2$$

For large imaging angle,  $qL \gg 1$

$$\langle |A_{\text{fr}}|^2 \rangle \sim L \rho^2 \xi_h \sin\left(\frac{\pi}{K}\right) \Gamma\left(1 - \frac{2}{K}\right) (\xi_h q)^{1/K-1}$$

Luttinger parameter  $K$  may be extracted from the **angular dependence** of  $A_{\text{fr}}(\theta)$

# Higher moments of interference amplitude



$A_{\text{fr}}$  is a quantum operator. The measured value of  $|A_{\text{fr}}|$  will fluctuate from shot to shot.

Can we predict the distribution function of  $|A_{\text{fr}}|$ ?

Higher moments

$$\langle |A_{\text{fr}}|^{2n} \rangle = \int_0^L dz_1 \dots dz'_n |\langle a^\dagger(z_1) \dots a^\dagger(z_n) a(z'_1) \dots a(z'_n) \rangle|^2$$

Changing to periodic boundary conditions (long condensates)

$$\langle |A_{\text{fr}}|^{2n} \rangle = \langle |A_{\text{fr}}|^2 \rangle^n \times Z_{2n}$$


$$Z_{2n} = \prod_{ij} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{du_i}{2\pi} \frac{dv_j}{2\pi} \left| \frac{\prod_{i<j} 2 \sin(\frac{u_i - u_j}{2}) \prod_{i<j} 2 \sin(\frac{v_i - v_j}{2})}{\prod_{ij} 2 \sin(\frac{u_i - v_j}{2})} \right|^{1/K}$$

Explicit expressions for  $Z_{2n}$  are available but cumbersome

Fendley, Lesage, Saleur, J. Stat. Phys. 79:799 (1995)

# Full counting statistics of interference experiments

## Impurity in a Luttinger liquid



$$S = \frac{\pi K}{2} \int dx d\tau [(\partial_\tau \phi)^2 + (\partial_x \phi)^2] + 2g \int d\tau \cos \phi(x=0, \tau)$$

Expansion of the partition function in powers of  $g$

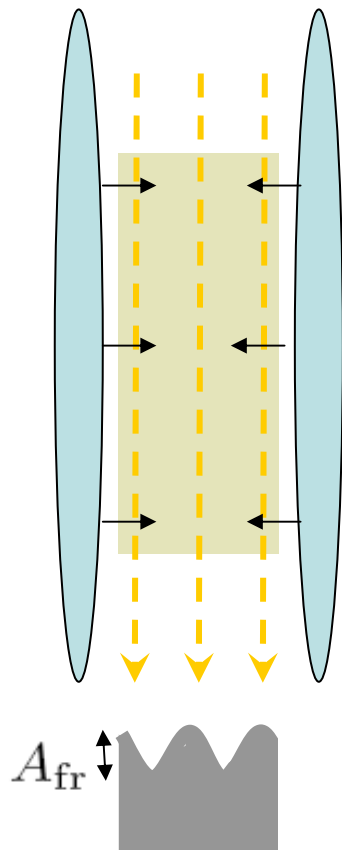
$$Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(2n)!} \int d\tau_1 \dots d\tau_n (e^{i\phi} + e^{-i\phi})_{\tau_1} \dots (e^{i\phi} + e^{-i\phi})_{\tau_{2n}}$$

$$Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(n!)^2} Z_{2n}$$

$$Z_{2n} = \prod_{ij} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{du_i}{2\pi} \frac{dv_j}{2\pi} \left| \frac{\prod_{i<j} 2 \sin(\frac{u_i - u_j}{2}) \prod_{i<j} 2 \sin(\frac{v_i - v_j}{2})}{\prod_{ij} 2 \sin(\frac{u_i - v_j}{2})} \right|^{1/K}$$

Partition function of the impurity contains correlation functions taken at the same point and at different times. Moments of interference experiments come from correlations functions taken at the same time but in different points. Lorentz invariance ensures that the two are the same

# Relation between quantum impurity problem and interference of fluctuating condensates



Normalized amplitude  
of interference fringes

$$a^2 = |A_{\text{fr}}|^2 / \langle |A_{\text{fr}}|^2 \rangle$$

Distribution function  
of fringe amplitudes

$$W(K, a^2)$$

Relation to the impurity partition function

$$Z_{\text{imp}}(K, g) = \int_0^\infty da^2 W(K, a^2) I_0(2g a)$$

Distribution function can be reconstructed from  $Z_{\text{imp}}(K, g)$   
using completeness relations for Bessel functions

$$W(K, a^2) = 2 \int_0^\infty g dg Z_{\text{imp}}(K, ig) J_0(2ga^2)$$

## Bethe ansatz solution for a quantum impurity

$Z_{\text{imp}}(K, g)$  can be obtained from the Bethe ansatz following

Zamolodchikov, Phys. Lett. B 253:391 (91); Fendley, et al., J. Stat. Phys. 79:799 (95)

Making analytic continuation is possible but cumbersome

## Interference amplitude and spectral determinant

$Z_{\text{imp}}(K, ig)$  is related to a Schroedinger equation

Dorey, Tateo, J.Phys. A. Math. Gen. 32:L419 (1999)

Bazhanov, Lukyanov, Zamolodchikov, J. Stat. Phys. 102:567 (2001)

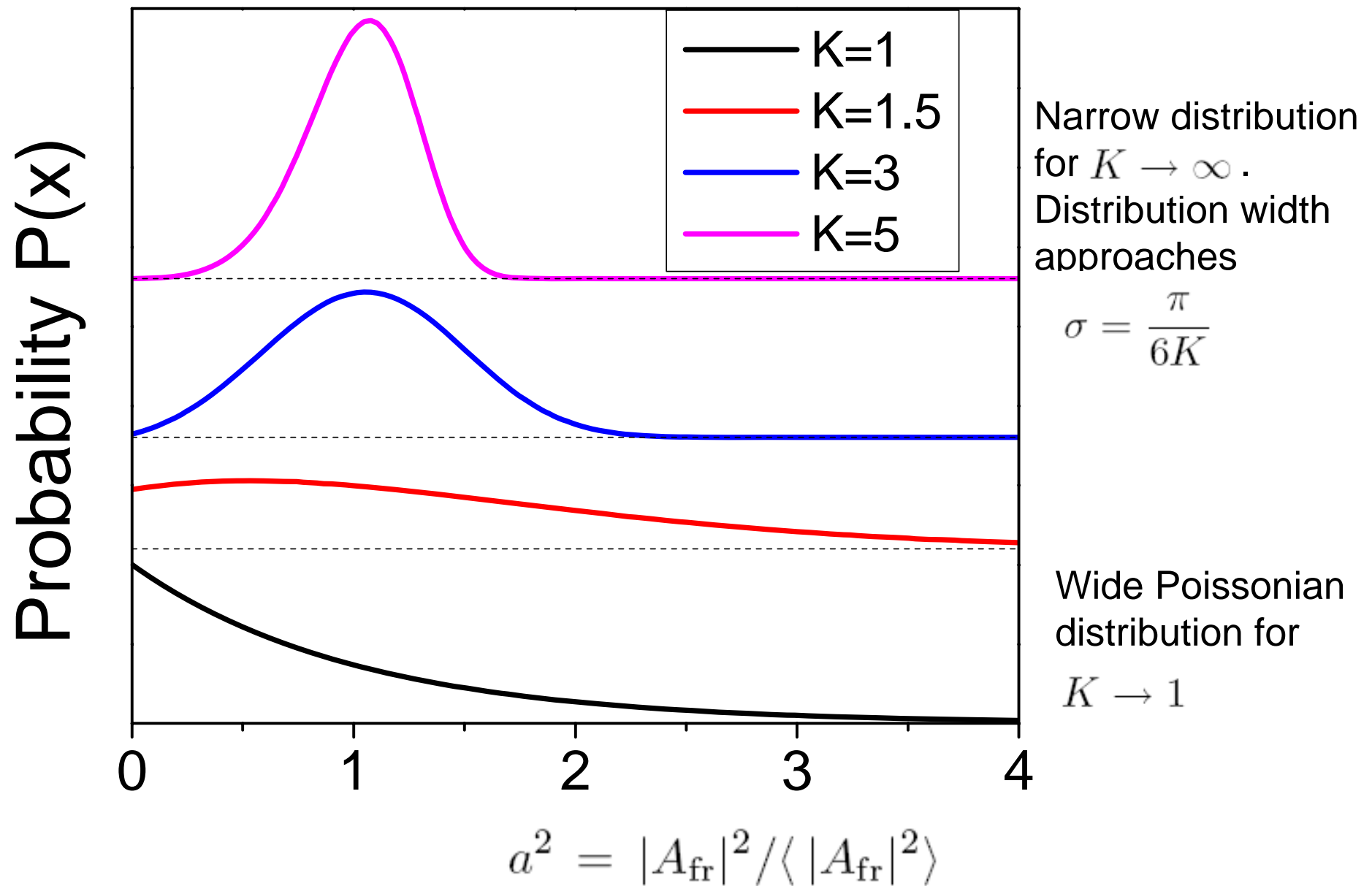
$$-\frac{d^2 \Psi}{dx^2} + \left( x^{4K-2} + \frac{3}{4x^2} \right) \Psi = E \Psi$$

Spectral determinant  $D(E) = \prod_{n=1}^{\infty} \left( 1 - \frac{E}{E_n} \right)$

$$Z_{\text{imp}}(K, ig) = D \left( \frac{g^2}{\pi^2} (4K)^{2-K^{-1}} \left[ \Gamma\left(1 - \frac{1}{2K}\right) \right]^2 \sin^2\left(\frac{\pi}{2K}\right) \right)$$



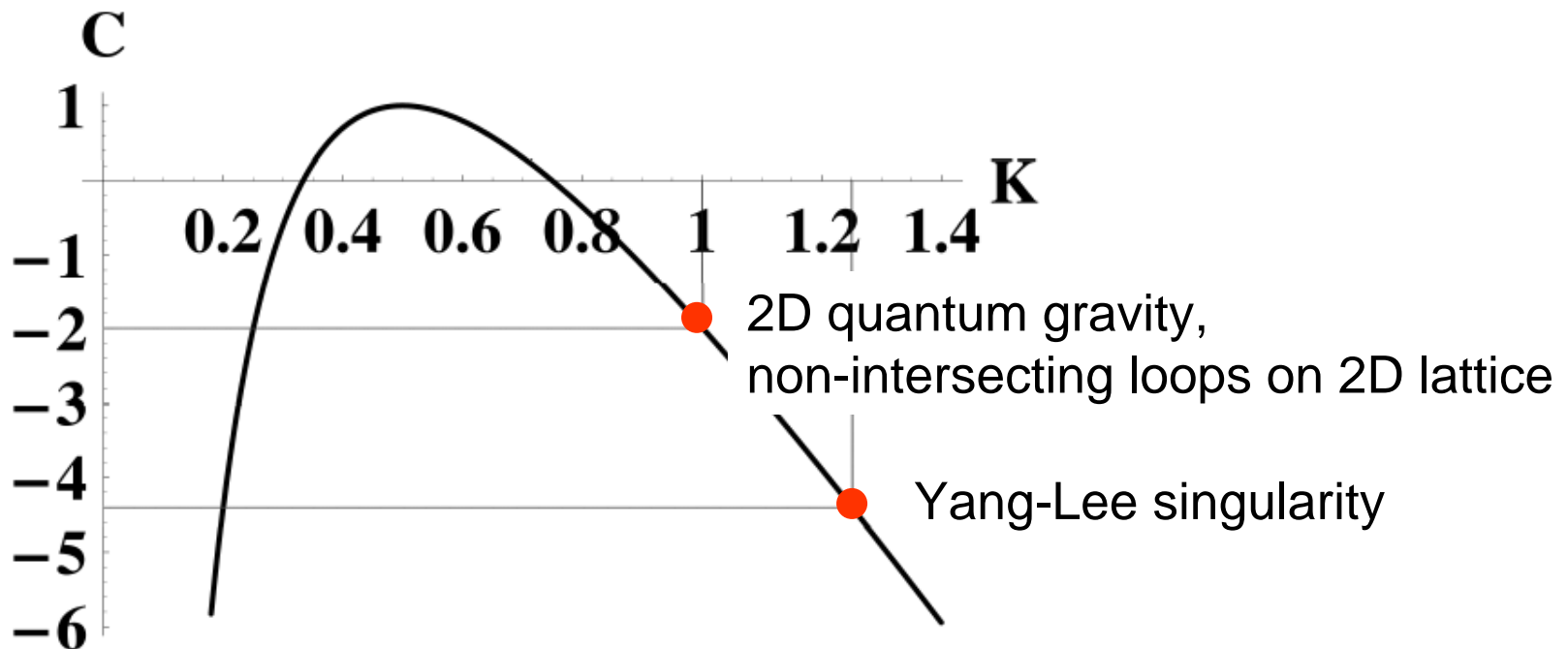
## Evolution of the distribution function



# From interference amplitudes to conformal field theories

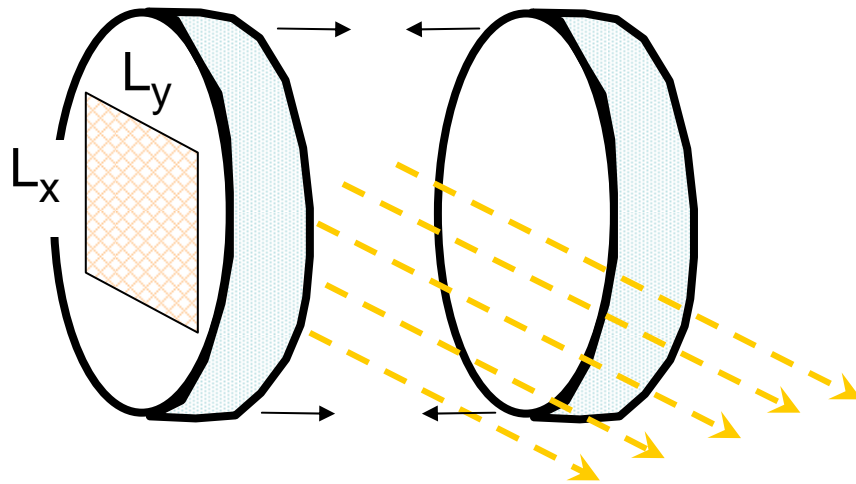
$Z_{\text{imp}}(K, ig)$  correspond to vacuum eigenvalues of  $Q$  operators of CFT  
Bazhanov, Lukyanov, Zamolodchikov, Comm. Math. Phys. 1996, 1997, 1999

When  $K > 1$ ,  $Z_{\text{imp}}(K, ig)$  is related to  $Q$  operators of CFT with  $c < 0$ . This includes 2D quantum gravity, non-intersecting loop model on 2D lattice, growth of random fractal stochastic interface, high energy limit of multicolor QCD, ...



# Interference of two dimensional condensates

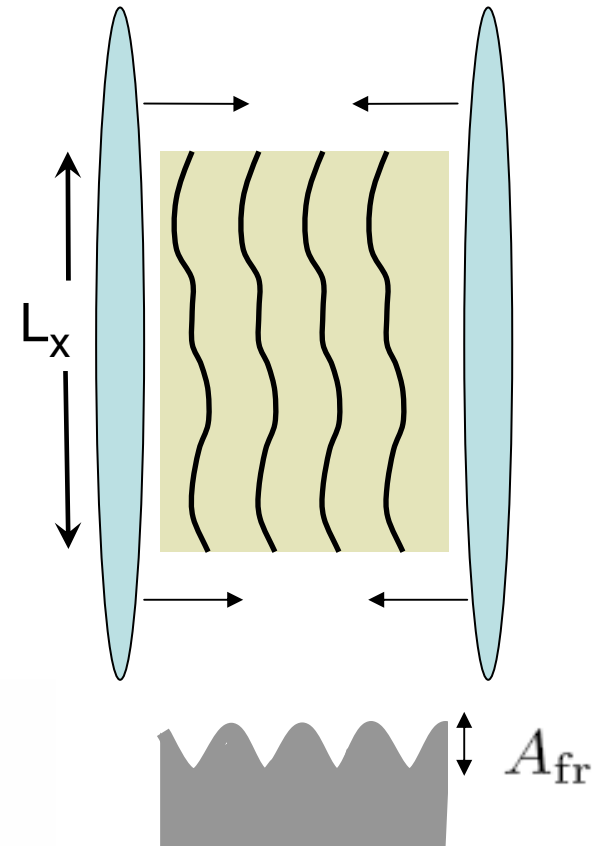
Experiments: Stock et al., cond-mat/0506559



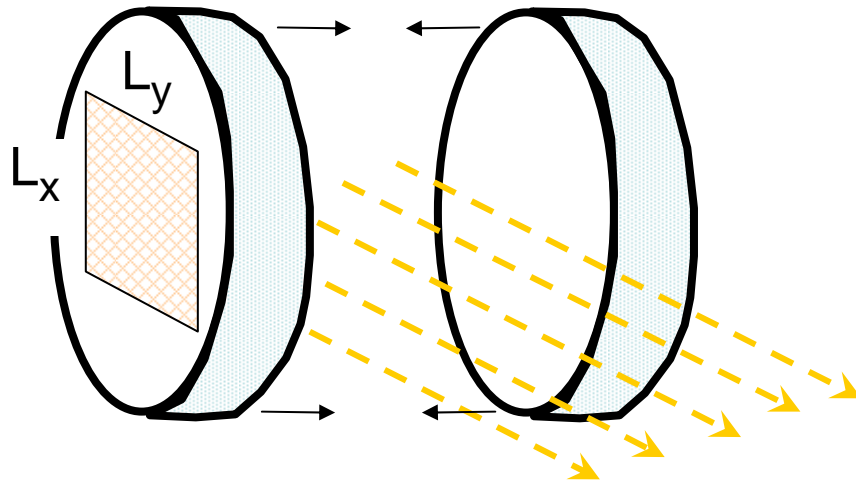
Probe beam parallel to the plane of the condensates

$$\langle |A_{\text{fr}}|^2 \rangle = L_x L_y \int_0^{L_x} \int_0^{L_y} d^2 \vec{r} (G(\vec{r}))^2$$

$$G(\vec{r}) = \langle a(\vec{r}) a^\dagger(0) \rangle$$



# Interference of two dimensional condensates. Quasi long range order and the KT transition



Above KT transition

$$G(r) \sim e^{-r/\xi}$$

$$\langle |A_{\text{fr}}|^2 \rangle \sim L_x L_y$$

$$\log \xi(T) \sim 1/\sqrt{T - T_{\text{KT}}}$$

Below KT transition

$$G(r) \sim \rho \left( \frac{\xi_h}{r} \right)^\alpha$$

$$\alpha(T) = \frac{mT}{2\pi\rho_s(T)\hbar^2}$$

$$\langle |A_{\text{fr}}|^2 \rangle \sim (L_x L_y)^{2-\alpha}$$

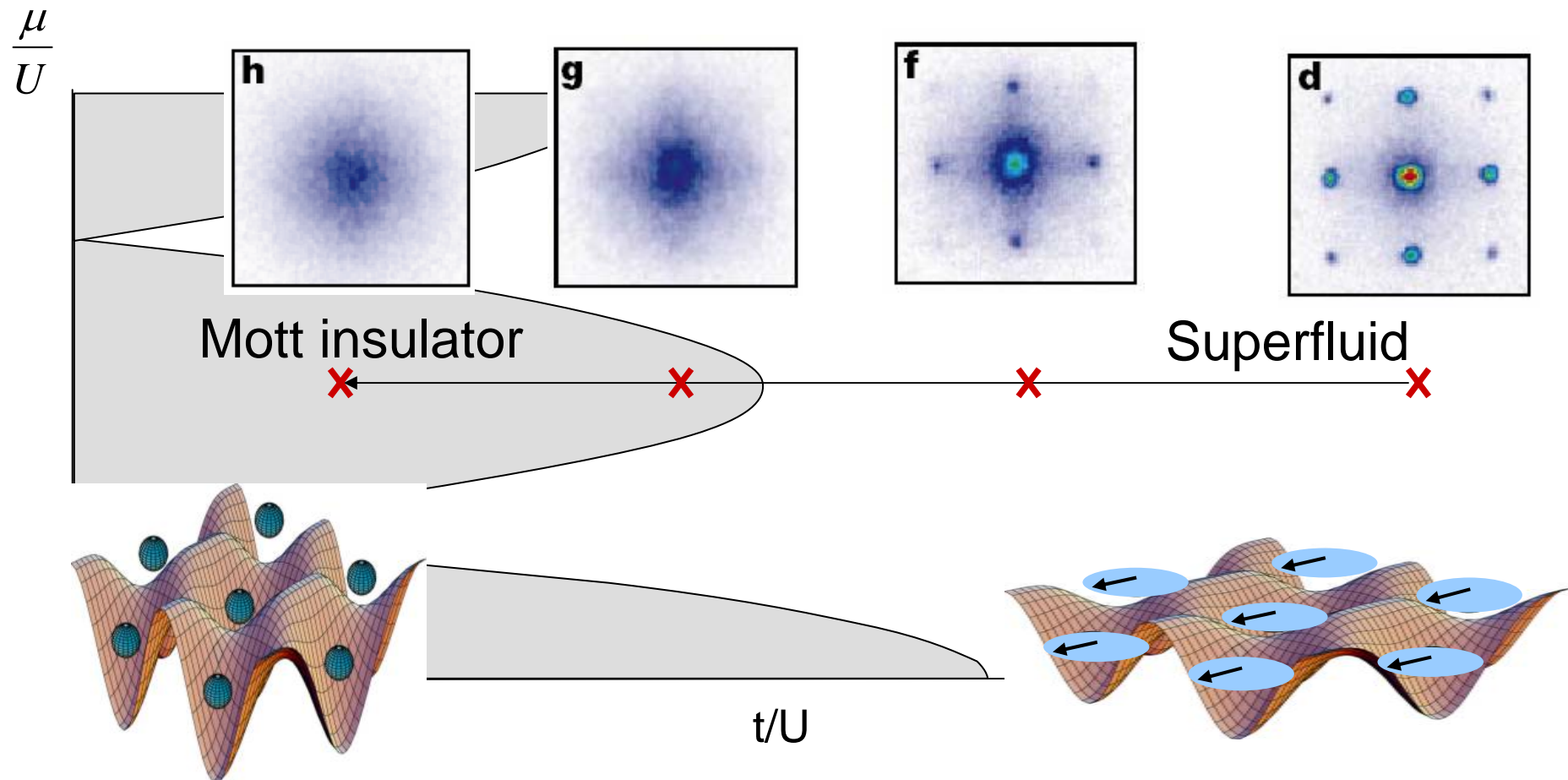
One can also use rotated probe beam experiments to extract  $\alpha$  from the angular dependence of  $A_{\text{fr}}$

# Quantum noise interferometry in time of flight experiments

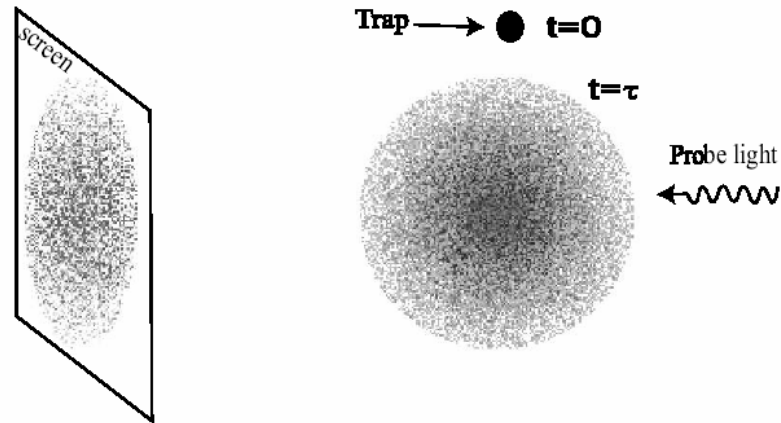
# Atoms in an optical lattice.

## Superfluid to Insulator transition

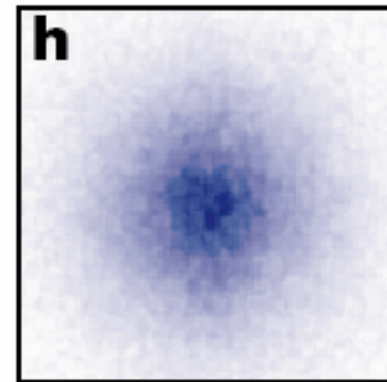
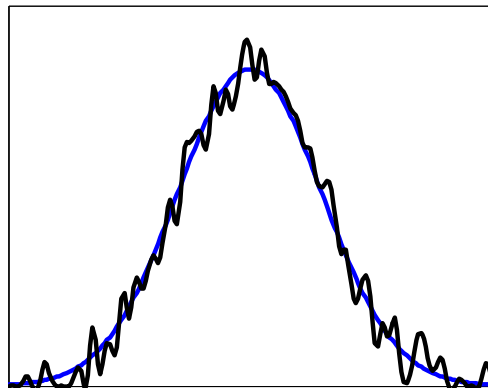
Greiner et al., Nature 415:39 (2002)



## Time of flight experiments



## Quantum noise interferometry of atoms in an optical lattice

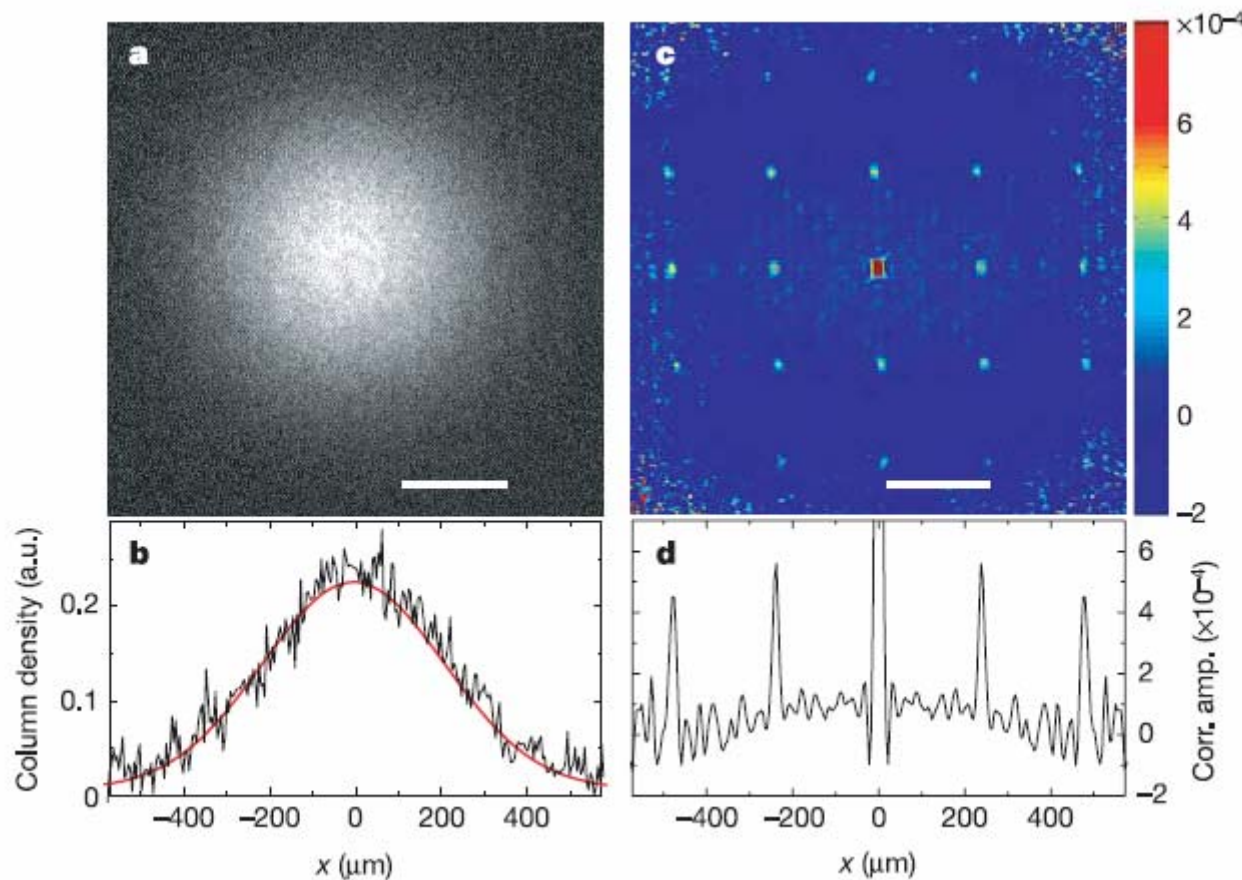


Second order coherence  $G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle$

# Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment

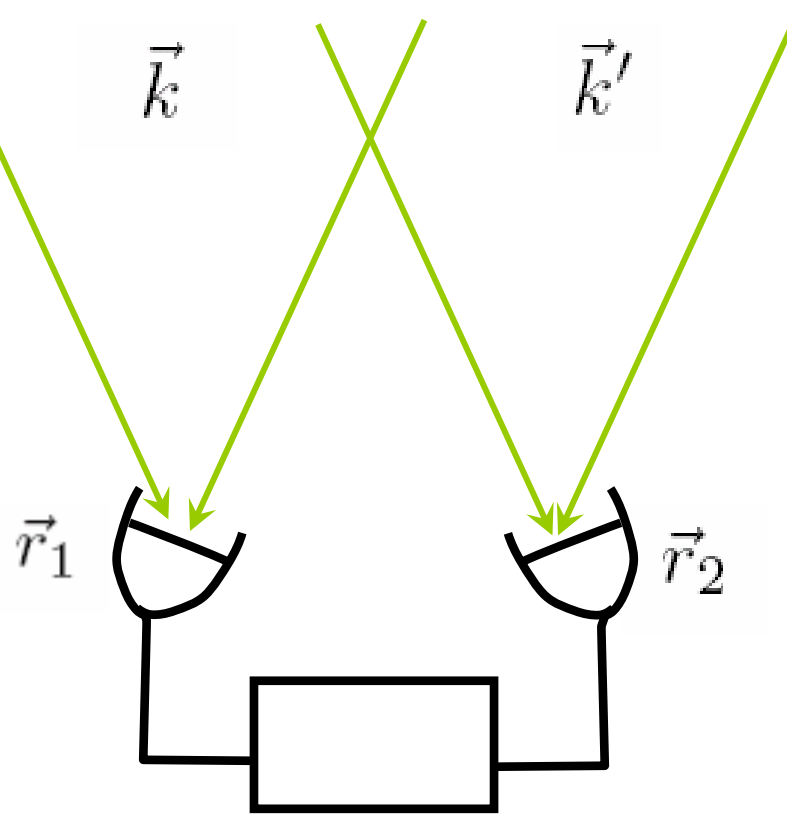
Theory: Altman et al., PRA 70:13603 (2004)

Experiment: Folling et al., Nature 434:481 (2005)



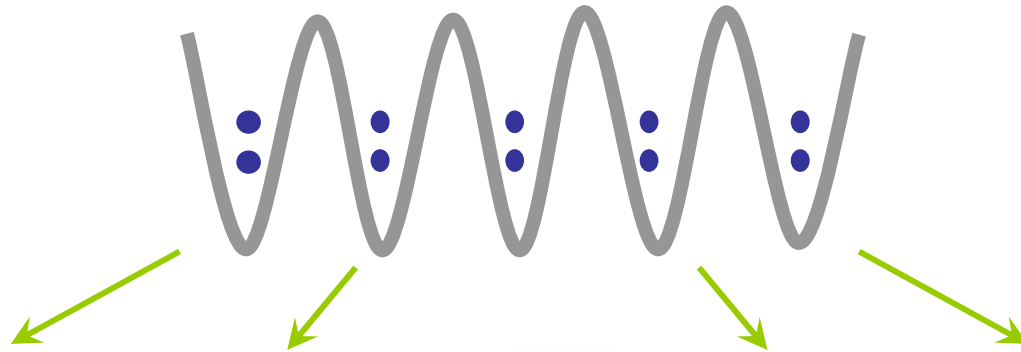


# Hanbury-Brown-Twiss stellar interferometer



$$\langle I(\vec{r}_1) I(\vec{r}_2) \rangle = A + B \cos \left( (\vec{k} - \vec{k}') \cdot (\vec{r}_1 - \vec{r}_2) \right)$$

## Second order coherence in the insulating state of bosons



Bosons at quasimomentum  $\vec{k}$  expand as plane waves

with wavevectors  $\vec{k}, \vec{k} + \vec{G}_1, \vec{k} + \vec{G}_2$

**First order coherence:**  $\langle \rho(\vec{r}) \rangle$

Oscillations in density disappear after summing over  $\vec{k}$

**Second order coherence:**  $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle$

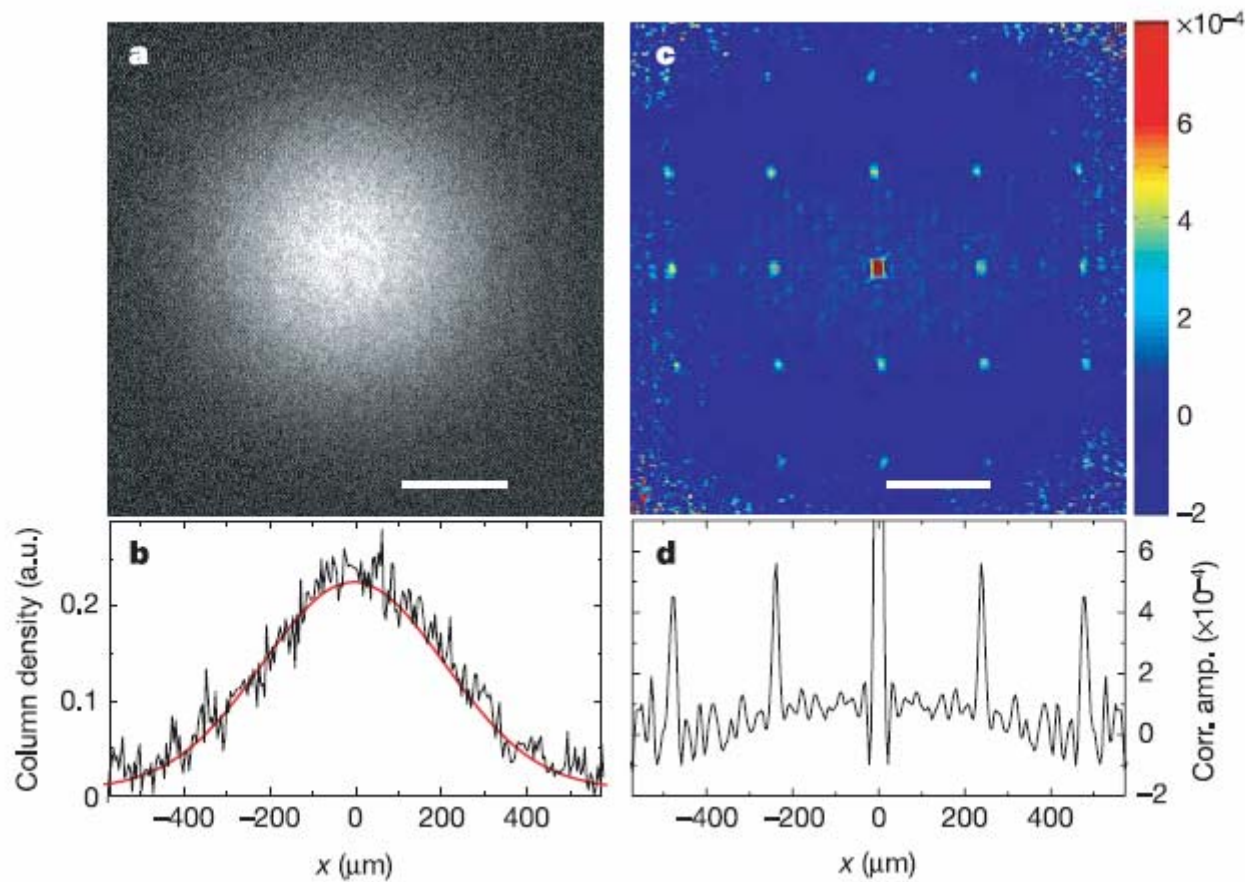
Correlation function acquires oscillations at reciprocal lattice vectors

$$\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle = A_0 + A_1 \cos \left( \vec{G}_1(\vec{r}_1 - \vec{r}_2) \right) + A_2 \cos \left( \vec{G}_2(\vec{r}_1 - \vec{r}_2) \right) + \dots$$

# Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment

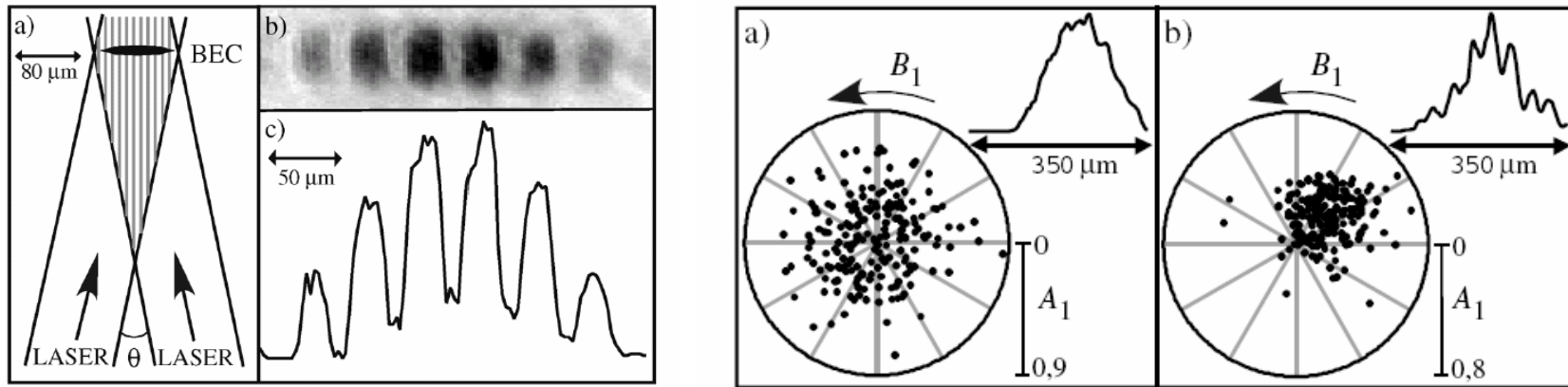
Theory: Altman et al., PRA 70:13603 (2004)

Experiment: Folling et al., Nature 434:481 (2005)

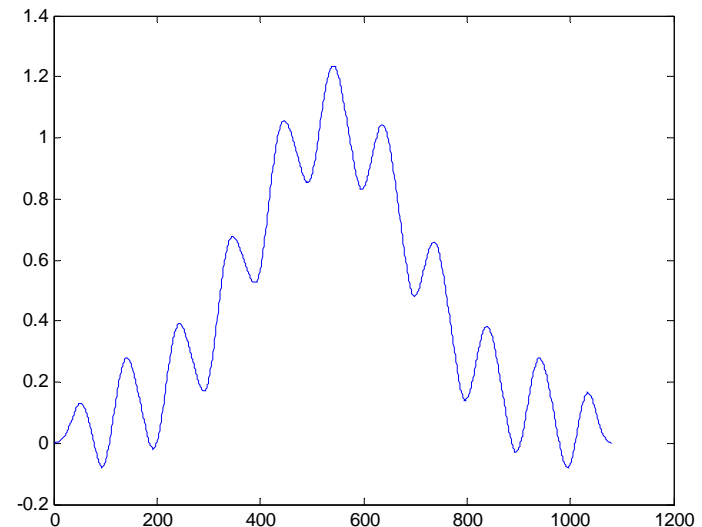
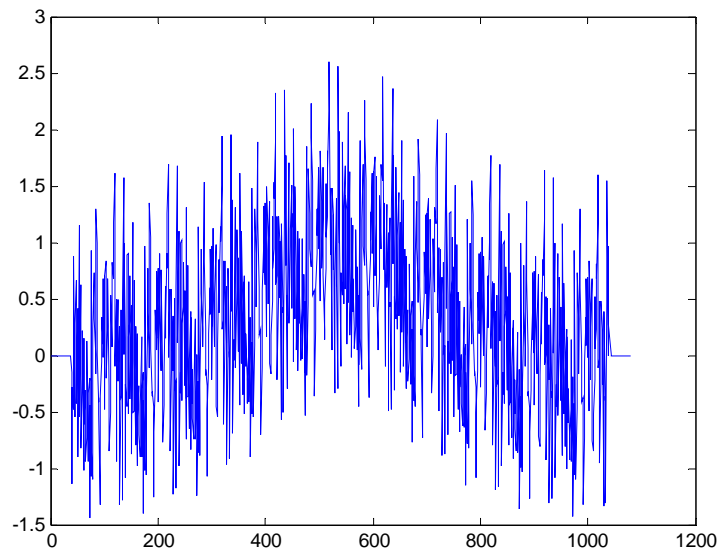


# Interference of an array of independent condensates

Hadzibabic et al., PRL 93:180403 (2004)



Smooth structure is a result of finite experimental resolution (filtering)

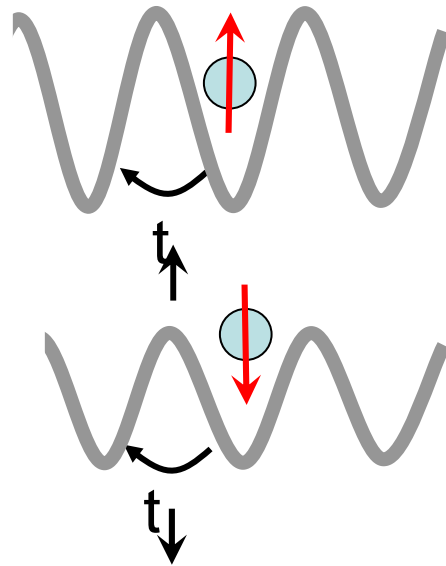


# Applications of quantum noise interferometry

Spin order in Mott states of atomic mixtures

# Two component Bose mixture in optical lattice

Example:  $^{87}\text{Rb}$ . Mandel et al., Nature 425:937 (2003)



$$|\uparrow\rangle = |F=1, m_F=-1\rangle$$

$$|\downarrow\rangle = |F=2, m_F=-2\rangle$$

Two component Bose Hubbard model

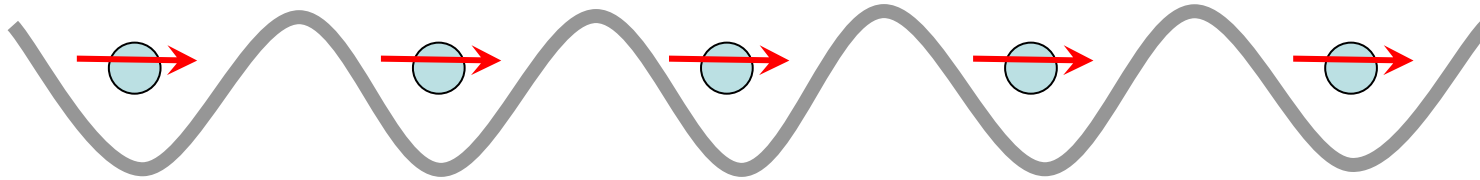
$$\begin{aligned} \mathcal{H} = & -t_{\uparrow} \sum_{\langle ij \rangle} b_{i\uparrow}^{\dagger} b_{j\uparrow} - t_{\downarrow} \sum_{\langle ij \rangle} b_{i\downarrow}^{\dagger} b_{j\downarrow} + U_{\uparrow\uparrow} \sum_i n_{i\uparrow}(n_{i\uparrow} - 1) \\ & + U_{\downarrow\downarrow} \sum_i n_{i\downarrow}(n_{i\downarrow} - 1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{i\downarrow} \end{aligned}$$

# Two component Bose mixture in optical lattice.

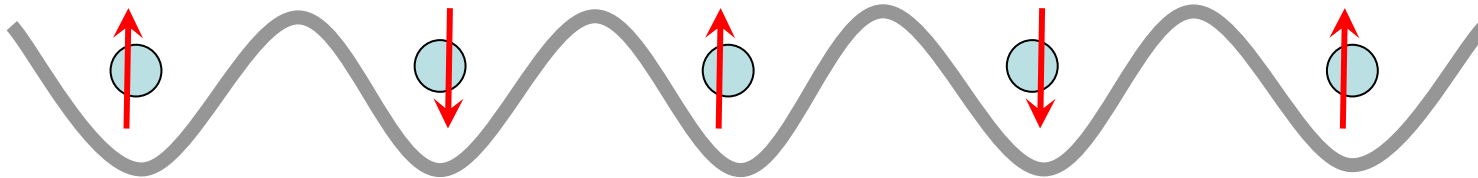
## Magnetic order in an insulating phase

Insulating phases with  $N=1$  atom per site. Average densities  $n_{\uparrow} = n_{\downarrow} = \frac{1}{2}$

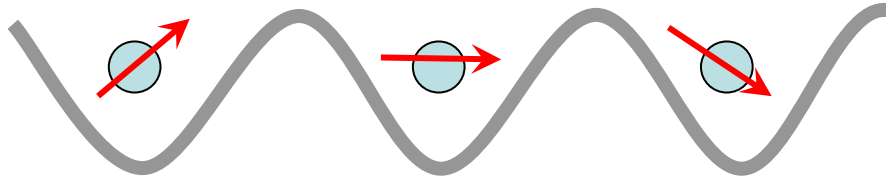
Easy plane ferromagnet  $|\Psi\rangle = \prod_i \left( b_{i\uparrow}^{\dagger} + e^{i\phi} b_{i\downarrow}^{\dagger} \right) |0\rangle$



Easy axis antiferromagnet  $|\Psi\rangle = \prod_{i \in A} b_{i\uparrow}^{\dagger} \prod_{i \in B} b_{i\downarrow}^{\dagger}$



# Quantum magnetism of bosons in optical lattices



Duan, Lukin, Demler, PRL (2003)

$$\mathcal{H} = J_z \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + J_{\perp} \sum_{\langle ij \rangle} ( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y )$$

$$J_z = \frac{t_{\uparrow}^2 + t_{\downarrow}^2}{2U_{\uparrow\downarrow}} - \frac{t_{\uparrow}^2}{U_{\uparrow\uparrow}} - \frac{t_{\downarrow}^2}{U_{\downarrow\downarrow}} \quad J_{\perp} = - \frac{t_{\uparrow} t_{\downarrow}}{U_{\uparrow\downarrow}}$$

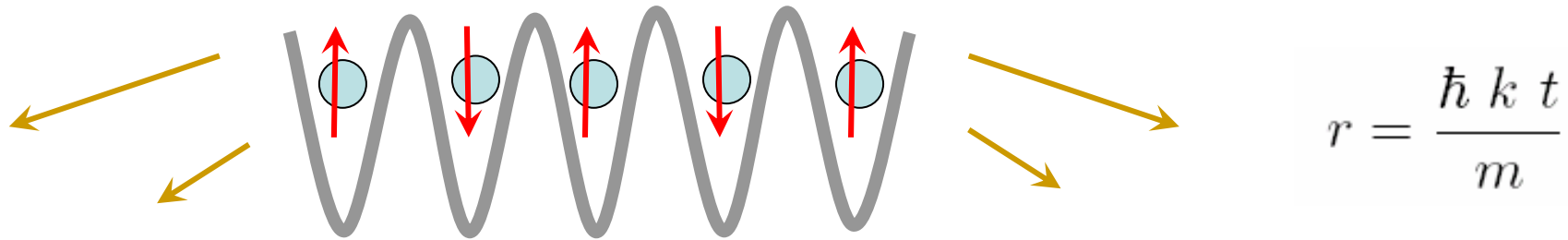
- Ferromagnetic
- Antiferromagnetic

$$U_{\uparrow\downarrow} \gg U_{\uparrow\uparrow}, U_{\downarrow\downarrow}$$

$$U_{\uparrow\downarrow} \ll U_{\uparrow\uparrow}, U_{\downarrow\downarrow}$$



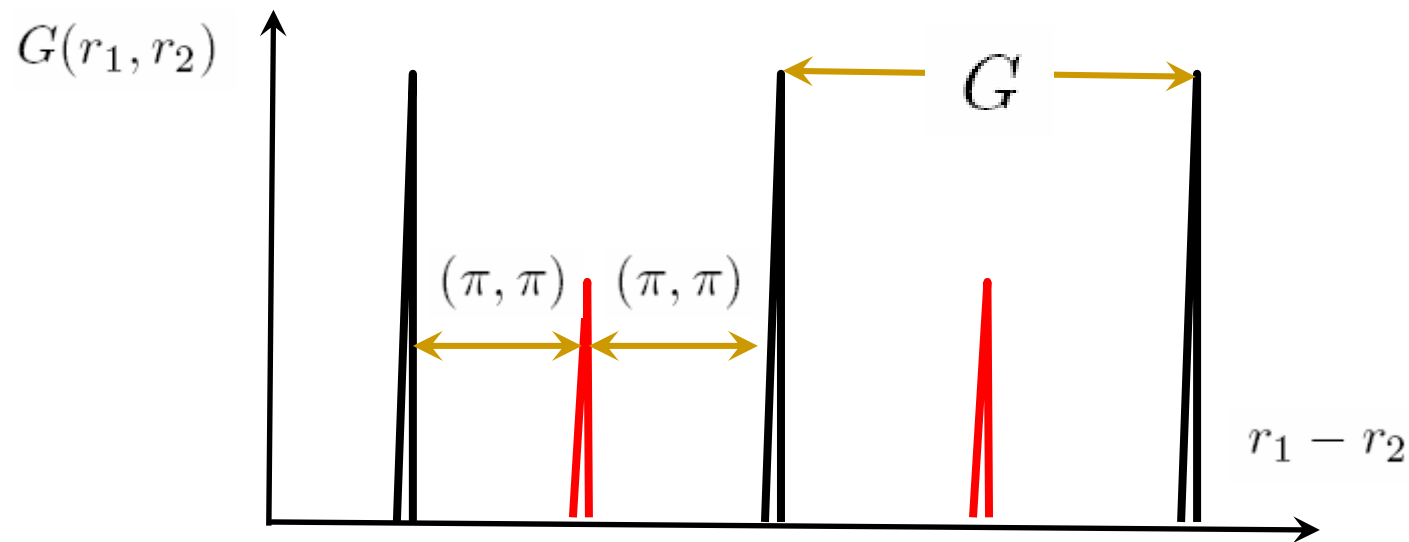
# Probing spin order of bosons



## Correlation Function Measurements

$$G(r_1, r_2) = \langle n(r_1) n(r_2) \rangle_{\text{TOF}} - \langle n(r_1) \rangle_{\text{TOF}} \langle n(r_2) \rangle_{\text{TOF}}$$

$$\sim \langle n(k_1) n(k_2) \rangle_{\text{LAT}} - \langle n(k_1) \rangle_{\text{LAT}} \langle n(k_2) \rangle_{\text{LAT}}$$



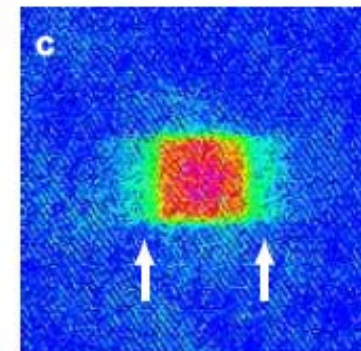
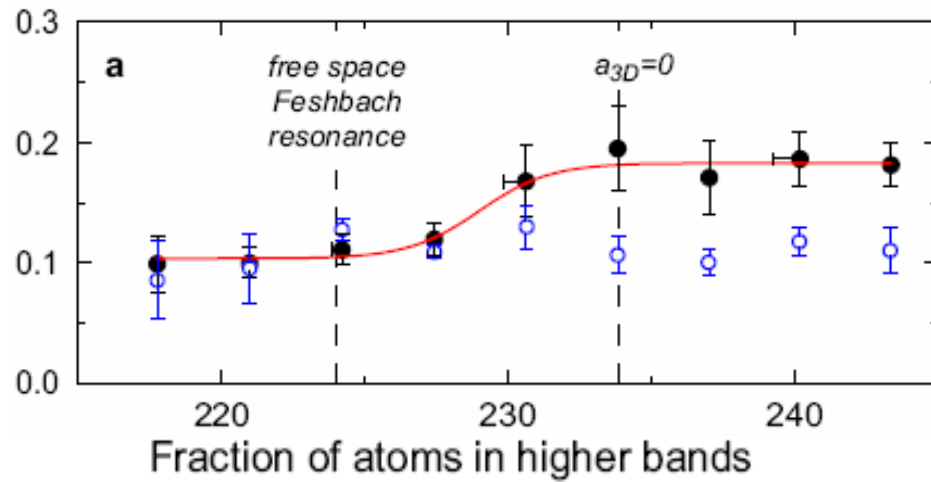
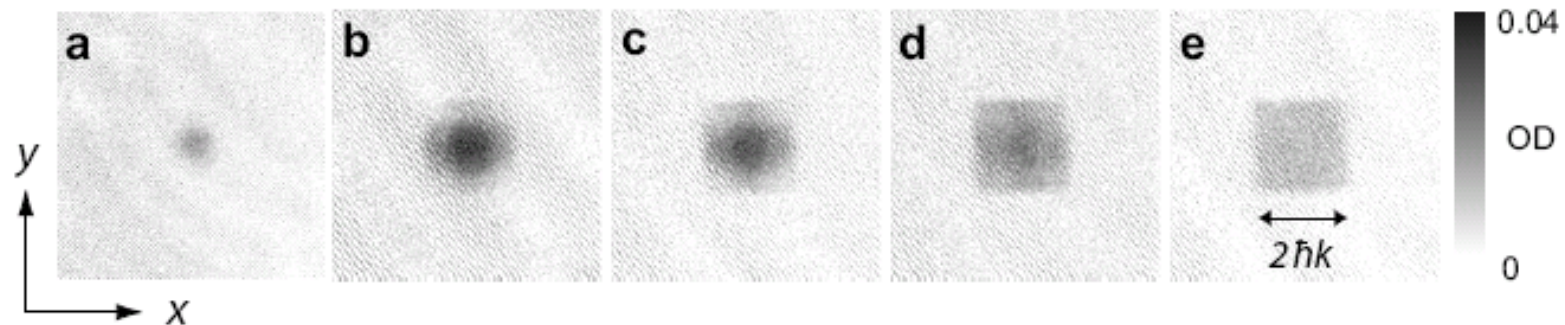
Extra Bragg peaks appear in the second order correlation function in the AF phase

# Applications of quantum noise interferometry

Detection of fermion pairing

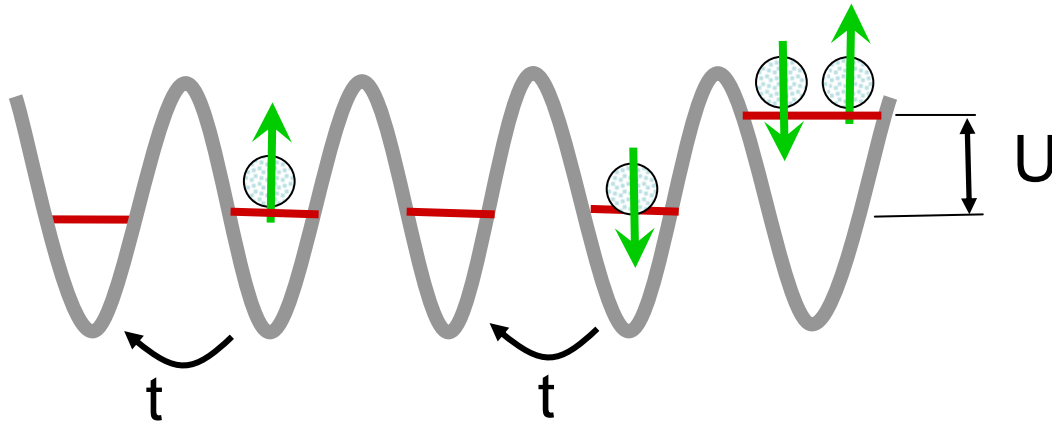
# Fermionic atoms in an optical lattice

Kohl et al., PRL 94:80403 (2005)

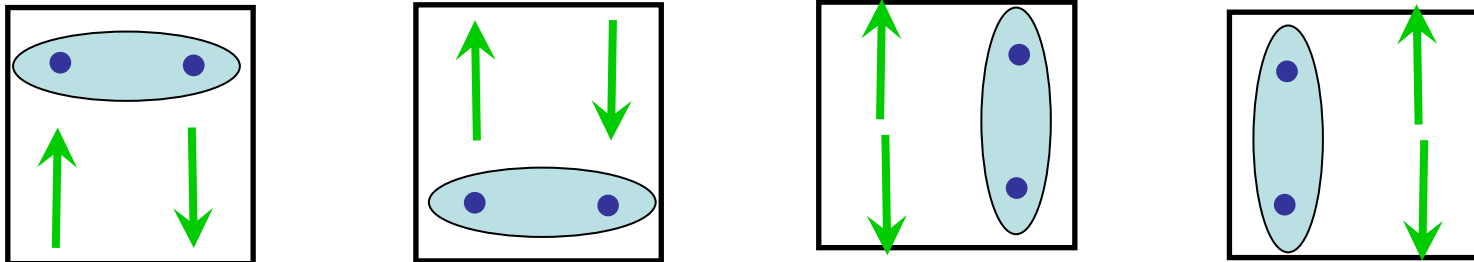


# Fermions with repulsive interactions

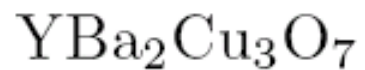
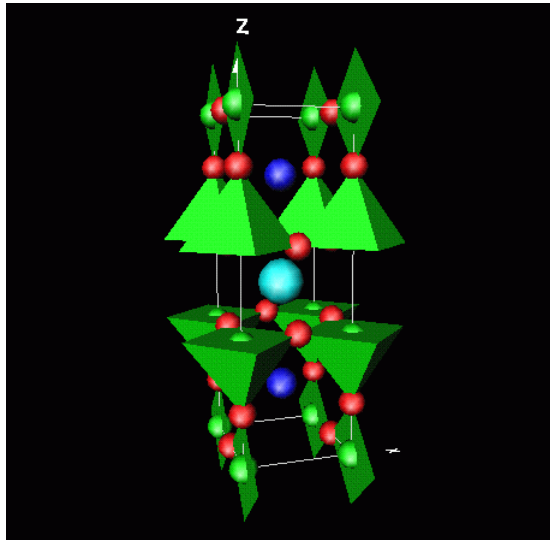
$$\mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_i n_i$$



**Possible d-wave pairing of fermions**

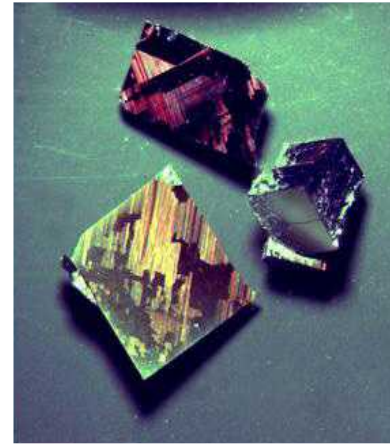


# High temperature superconductors



**Superconducting**

**T<sub>c</sub> 93 K**



Picture courtesy of UBC  
Superconductivity group

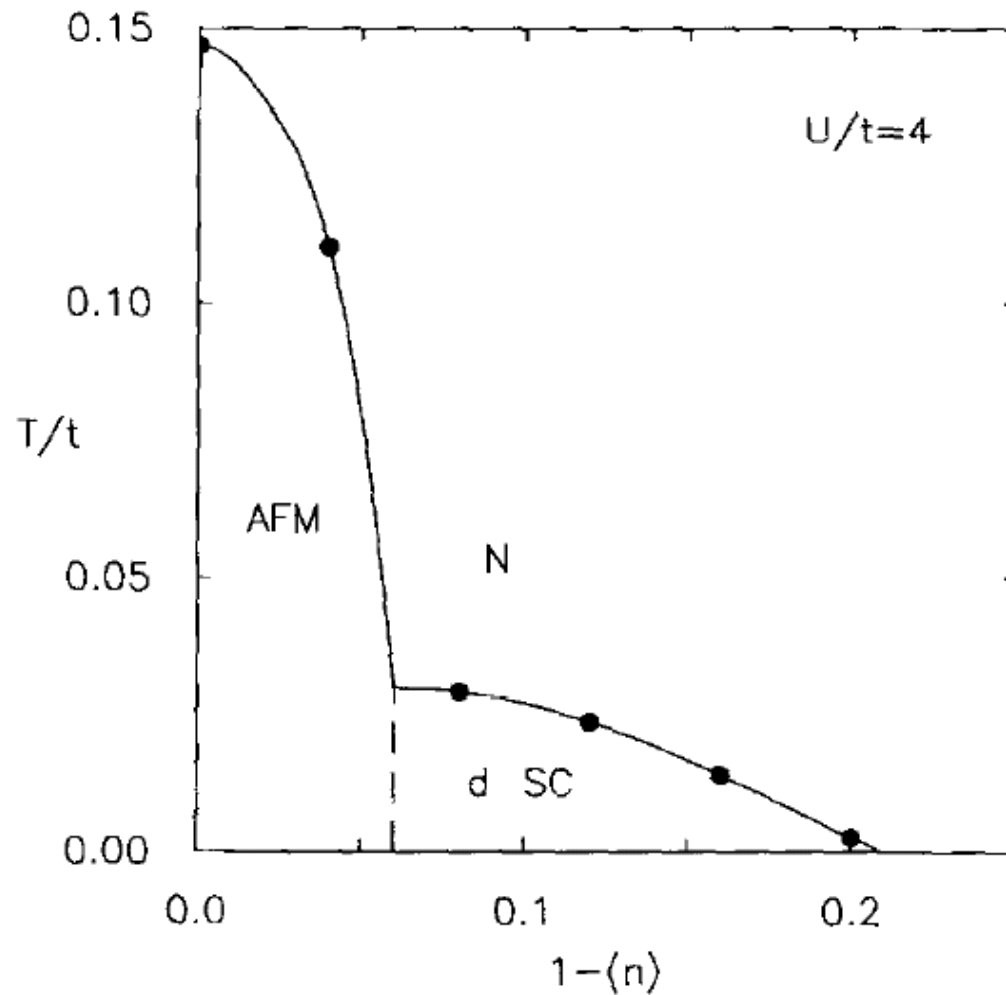
**Hubbard model** – minimal model for cuprate superconductors

P.W. Anderson, cond-mat/0201429

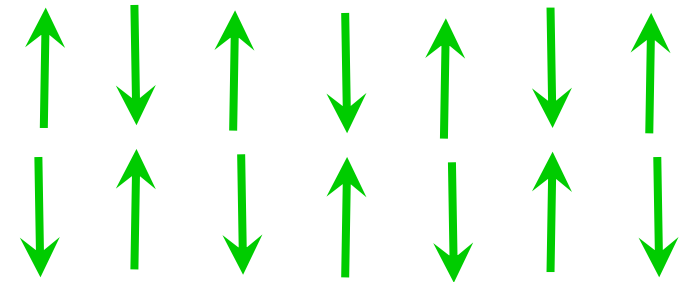
**After twenty years of work we still do not understand  
the fermionic Hubbard model**

# Positive U Hubbard model

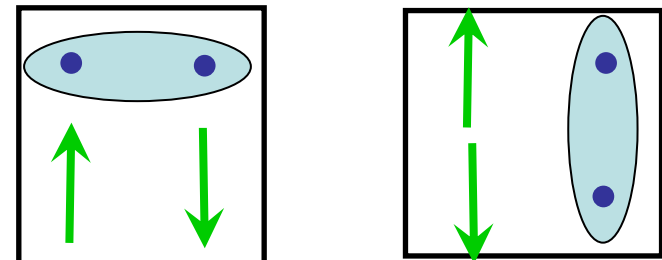
Possible phase diagram. Scalapino, Phys. Rep. 250:329 (1995)



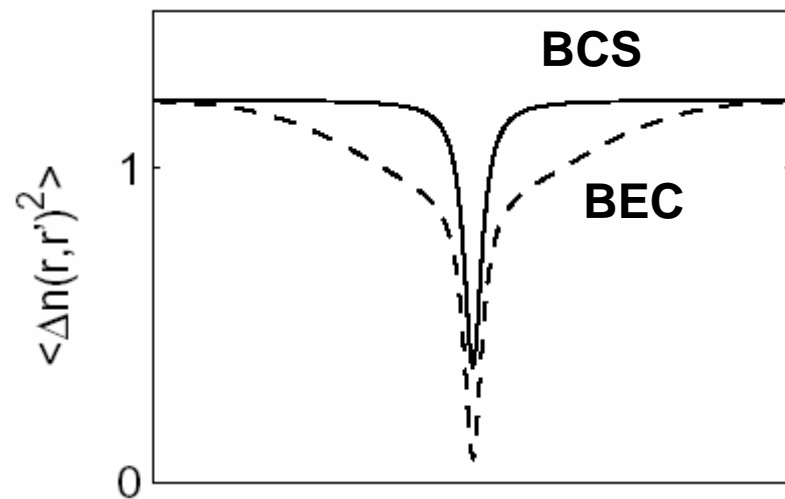
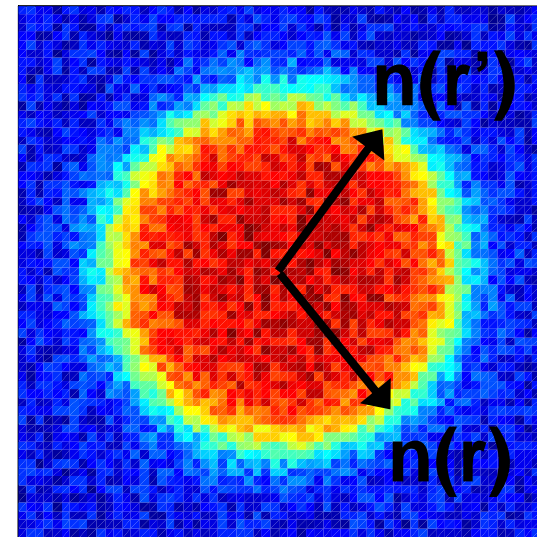
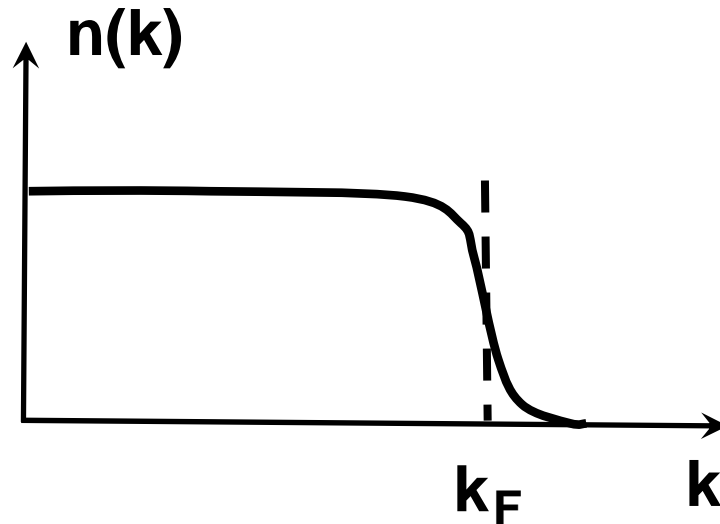
**Antiferromagnetic insulator**



**D-wave superconductor**



## Second order interference from a BCS superfluid



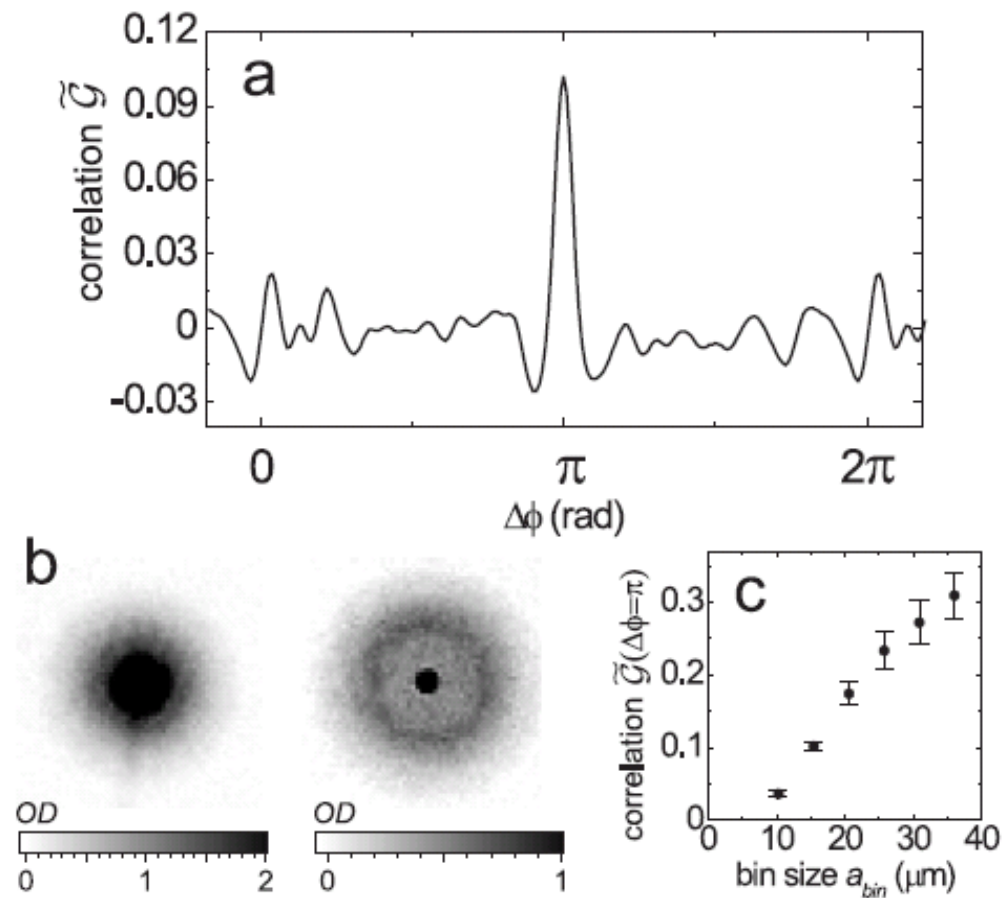
$$\Delta n(\mathbf{r}, \mathbf{r}') \equiv n(\mathbf{r}) - n(\mathbf{r}')$$

$$\Delta n(\mathbf{r}, -\mathbf{r}) | \Psi_{BCS} \rangle = 0$$

# Momentum correlations in paired fermions

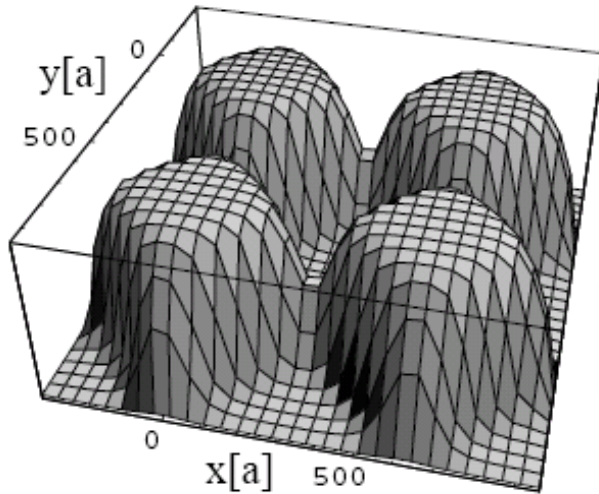
Theory: Altman et al., PRA 70:13603 (2004)

Experiment: Greiner et al., PRL 94:110401 (2005)





# Fermion pairing in an optical lattice



**Second Order Interference  
In the TOF images**

$$G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle$$

**Normal State**

$$G_N(r_1, r_2) = \delta(r_1 - r_2)\rho(r_1) - \rho^2(r_1) \sum_G \delta(r_1 - r_2 - \frac{G\hbar t}{m})$$

**Superfluid State**

$$G_S(r_1, r_2) = G_N(r_1, r_2) + \Psi(r_1) \sum_G \delta(r_1 + r_2 + \frac{G\hbar t}{m})$$

$\Psi(r) = |u(Q(r))v(Q(r))|^2$  measures the Cooper pair wavefunction

$$Q(r) = \frac{mr}{\hbar t}$$

**One can identify unconventional pairing**

# Conclusions

**We understand well:** electron systems in semiconductors and simple metals. Interaction energy is smaller than the kinetic energy. Perturbation theory works

**We do not understand:** strongly correlated electron systems in novel materials. Interaction energy is comparable or larger than the kinetic energy. Many surprising new phenomena occur, including high temperature superconductivity, magnetism, fractionalization of excitations

Ultracold atoms have energy scales of  $10^{-6}$ K, compared to  $10^4$  K for electron systems. However, **by engineering and studying strongly interacting systems of cold atoms we should get insights into the mysterious properties of novel quantum materials**

Our big goal is to develop a general framework for understanding strongly correlated systems. This will be important far beyond AMO and condensed matter

Developing new detection methods is an important problem in the area of strongly correlated atoms. **Interference experiments** and analysis of **quantum noise in time of flight experiments** are powerful tools for analyzing many-body states