Measuring correlation functions in interacting systems of cold atoms

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Correlation functions in condensed matter physics

Most experiments in condensed matter physics measure correlation functions.

Example: neutron scattering measures spin and density correlation functions.

\[ S_s(q) = \int d\mathbf{r} \ e^{i\mathbf{q}\cdot\mathbf{r}} \langle S^+(\mathbf{r}) S^-(0) \rangle \]
\[ S_\rho(q) = \int d\mathbf{r} \ e^{i\mathbf{q}\cdot\mathbf{r}} \langle \rho(\mathbf{r}) \rho(0) \rangle \]

Neutron diffraction patterns for MnO

Shull et al., Phys. Rev. 83:333 (1951)
Outline

Lecture I:
Measuring correlation functions in interference experiments

Lecture II:
Quantum noise interferometry in time of flight experiments

Emphasis of these lectures:
detection and characterization of many-body quantum states
Lecture I

Measuring correlation functions in interference experiments

1. Interference of independent condensates
2. Interference of interacting 1D systems
3. Interference of 2D systems
4. Full distribution function of the fringe amplitudes in interference experiments.
5. Studying coherent dynamics of strongly interacting systems in interference experiments
Lecture II

Quantum noise interferometry in time of flight experiments

1. Detection of spin order in Mott states of atomic mixtures
2. Detection of fermion pairing
Measuring correlation functions in interference experiments

Analysis of high order correlation functions in low dimensional systems

Polkovnikov, Altman, Demler, PNAS (2006)
Interference of two independent condensates

Interference of two independent condensates

Clouds 1 and 2 do not have a well defined phase difference. However each individual measurement shows an interference pattern

\[
\rho_{\text{int}}(r) = a_1^\dagger(r) a_2(r) + \text{c.c.}
\]

\[
a_1(r) = e^{i \phi_1 + i k_1 r} \quad k_1 = \frac{m v_1}{\hbar} = \frac{mr}{\hbar t}
\]

\[
a_2(r) = e^{i \phi_2 + i k_2 r} \quad k_2 = \frac{m v_2}{\hbar} = \frac{m(r + d)}{\hbar t}
\]

\[
\rho_{\text{int}}(r) = e^{i (k_2 - k_1) r} e^{i (\phi_2 - \phi_1)} + \text{c.c.}
\]

\[
\rho_{\text{int}}(r) = e^{i \frac{m d r}{\hbar t}} e^{i (\phi_2 - \phi_1)} + \text{c.c.}
\]

\[
\langle \rho_{\text{int}}(r) \rangle = 0
\]

\[
\langle \rho_{\text{int}}(r) \rho_{\text{int}}(r') \rangle = e^{i \frac{m d}{\hbar t} (r - r')} + \text{c.c.}
\]
Interference of one dimensional condensates


Amplitude of interference fringes, $A_{\text{fr}}$, contains information about phase fluctuations within individual condensates

$$d\rho_{\text{int}} (x, y) = (e^{i \frac{mdy}{\hbar t}} a_1^\dagger (x) a_2 (x) + \text{c.c.}) \, dx$$

$$\rho_{\text{int}} (y) = e^{i \frac{mdy}{\hbar t}} \int_0^L dx \, a_1^\dagger (x) a_2 (x) + \text{c.c.}$$

$$\rho_{\text{int}} (y) = A_{\text{fr}} e^{i \Delta \phi + i \frac{mdy}{\hbar t}} + \text{c.c.}$$
Interference amplitude and correlations

\[ A_{fr} e^{i \Delta \phi} = \int_0^L dx \ a_1^\dagger(x) a_2(x) \]

\[ \langle |A_{fr}|^2 \rangle = \int_0^L \int_0^L dx \, dy \ \langle a_1^\dagger(x) a_2(x) a_2^\dagger(y) a_1(y) \rangle \]

\[ \approx L \int_0^L dx \ \langle a_1(x) a_1^\dagger(0) \rangle \langle a_2(0) a_2^\dagger(x) \rangle \]

For identical condensates

\[ \langle |A_{fr}|^2 \rangle = L \int_0^L dx \ (G(x))^2 \]

Instantaneous correlation function

\[ G(x) = \langle a(x) a^\dagger(0) \rangle \]
Interacting bosons in 1d at T=0

Low energy excitations and long distance correlation functions can be described by the Luttinger Hamiltonian.

\[ \mathcal{H} = \int dx \, d\tau \left[ \frac{1}{2K} n^2 + \frac{K}{2} (\partial_x \phi)^2 \right] \]

\[ [n(x_1), \phi(x_2)] = -i \, \delta(x_1 - x_2) \]

\( K \) – Luttinger parameter

Connection to original bosonic particles

\[ a(x) \sim e^{i\phi(x)} \]

Small \( K \) corresponds to strong quantum fluctuations
Luttinger liquids in 1d

For non-interacting bosons

\[ K \rightarrow \infty \]

\[ \mathcal{H} = \int dx \, d\tau \left[ \frac{1}{2K} n^2 + \frac{K}{2} (\partial_x \phi)^2 \right] \]

\[ K = \infty \]

For impenetrable bosons

\[ K \rightarrow 1 \]

\[ G(x) \sim \rho \left( \frac{\xi_h}{x} \right)^{1/2K} \]

Correlation function decays rapidly for small K. This decay comes from strong quantum fluctuations
Interference between 1d interacting bosons

Luttinger liquid at T=0

\[ G(x) \sim \rho \left( \frac{\xi_h}{x} \right)^{1/2K} \]

\( K \) – Luttinger parameter

\[ \langle |A_{fr}|^2 \rangle \sim (\rho \xi_h)^{1/K} (L\rho)^{2-1/K} \]

For non-interacting bosons

\( K = \infty \) and \( A_{fr} \sim L \)

For impenetrable bosons

\( K = 1 \) and \( A_{fr} \sim \sqrt{L} \)

Luttinger liquid at finite temperature

\[ \langle |A_{fr}|^2 \rangle \sim L \rho^2 \xi_h \left( \frac{\hbar^2}{m \xi_h^2} \frac{1}{T} \right)^{1-1/K} \]

Analysis of \( A_{fr} \) can be used for thermometry
Rotated probe beam experiment

\[ \langle |A_{fr}|^2 \rangle = L \int_0^L dx \cos(qx) (G(x))^2 \]

For large imaging angle, \( qL \gg 1 \)

\[ \langle |A_{fr}|^2 \rangle \sim L \rho^2 \xi_h \sin\left( \frac{\pi}{K} \right) \Gamma\left( 1 - \frac{2}{K} \right) (\xi_h q)^{1/K-1} \]

Luttinger parameter \( K \) may be extracted from the angular dependence of \( A_{fr}(\theta) \)

\[ q = \frac{md}{\hbar t} \tan \theta \]
Interference between two-dimensional BECs at finite temperature. Kosteritz-Thouless transition
Interference of two dimensional condensates

Experiments: Stock, Hadzibabic, Dalibard, et al., cond-mat/0506559
Gati, Oberthaler, et al., cond-mat/0601392

Probe beam parallel to the plane of the condensates

\[
\langle |A_{fr}|^2 \rangle = L_x L_y \int_0^{L_x} \int_0^{L_y} d^2 \vec{r} \left( G(\vec{r}) \right)^2
\]

\[
G(\vec{r}) = \langle a(\vec{r}) a^\dagger(0) \rangle
\]
Interference of two dimensional condensates. Quasi long range order and the KT transition

Above Kosterlitz-Thouless transition: Vortices proliferate. Short range order

Below Kosterlitz-Thouless transition: Vortices confined. Quasi long range order

Above KT transition

\[ G(r) \sim e^{-r/\xi} \]
\[ \langle |A_{fr}|^2 \rangle \sim L_x L_y \]
\[ \log \xi(T) \sim 1/\sqrt{T - T_{KT}} \]

Below KT transition

\[ G(r) \sim \rho \left( \frac{\xi_h}{r} \right)^\alpha \]
\[ \alpha(T) = \frac{mT}{2\pi \rho_s(T) \hbar^2} \]
\[ \langle |A_{fr}|^2 \rangle \sim (L_x L_y)^{2-\alpha} \]
Experiments with 2D Bose gas


Typical interference patterns

low temperature

higher temperature
Experiments with 2D Bose gas


Contrast after integration

integration over $x$ axis

integration over $x$ axis

integration over $x$ axis

Contrast after integration

middle $T$

low $T$

high $T$

integration distance $D_x$ (pixels)
Experiments with 2D Bose gas


fit by:

$$C^2 \sim \frac{1}{D_x} \int [g_1(0,x)]^2 dx \sim \left( \frac{1}{D_x} \right)^{2\alpha}$$

Exponent $\alpha$

- if $g_1(r)$ decays exponentially with $\ell_{coh} \ll D_x$: $\alpha = 1/2$
- if $g_1(r)$ decays algebraically or exponentially with a large $\ell_{coh}$: $\alpha < 1/2$
Experiments with 2D Bose gas


c.f. Bishop and Reppy

He experiments:
universal jump in
the superfluid density

Ultracold atoms experiments:
jump in the correlation function.
KT theory predicts $\alpha=1/4$
just below the transition
Experiments with 2D Bose gas. Proliferation of thermal vortices


Fraction of images showing at least one dislocation
Rapidly rotating two dimensional condensates

Time of flight experiments with rotating condensates correspond to density measurements

\[ \langle \rho(r) \rangle \]

\[ \langle \rho(r) \rho(r') \rangle \]

Interference experiments measure single particle correlation functions in the rotating frame

\[ \langle |A_{fr}|^2 \rangle = L_x L_y \int_0^{L_x} \int_0^{L_y} d^2\vec{r} \cos(\vec{q} \cdot \vec{r}) \left( G(\vec{r}) \right)^2 \]

\[ G(\vec{r}) = \langle a(\vec{r}) a^{\dagger}(0) \rangle \]
Interference between two interacting one dimensional Bose liquids

Full distribution function of the amplitude of interference fringes

Gritsev, Altman, Demler, Polkovnikov, cond-mat/0602475
Higher moments of interference amplitude

$A_{fr}$ is a quantum operator. The measured value of $|A_{fr}|$ will fluctuate from shot to shot. Can we predict the distribution function of $|A_{fr}|$?

Higher moments

$$\langle |A_{fr}|^{2n} \rangle = \int_0^L dz_1 \ldots dz_n' \langle a^\dagger(z_1) \ldots a^\dagger(z_n) a(z'_1) \ldots a(z'_n) \rangle^2$$

Changing to periodic boundary conditions (long condensates)

$$\langle |A_{fr}|^{2n} \rangle = \langle |A_{fr}|^2 \rangle^n \times Z_{2n}$$

$$Z_{2n} = \prod_{i,j} \int_0^{2\pi} \frac{2\pi}{2\pi} \prod_{i<j} \left| \frac{2 \sin\left( \frac{u_i - u_j}{2} \right) \prod_{i<j} 2 \sin\left( \frac{v_i - v_j}{2} \right)}{\prod_{ij} 2 \sin\left( \frac{u_i - v_j}{2} \right)} \right|^{1/K}$$

Explicit expressions for $Z_{2n}$ are available but cumbersome

Impurity in a Luttinger liquid

$$S = \frac{\pi K}{2} \int dx \, d\tau \left[ (\partial_x \phi)^2 + (\partial_x \phi)^2 \right] + 2g \int d\tau \cos \phi (x = 0, \tau)$$

Expansion of the partition function in powers of $g$

$$Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(2n)!} \int d\tau_1 \cdots d\tau_n (e^{i\phi} + e^{-i\phi})_{\tau_1} \cdots (e^{i\phi} + e^{-i\phi})_{\tau_{2n}}$$

$$Z_{\text{imp}} = \sum \frac{g^{2n}}{(n!)^2} Z_{2n}$$

$$Z_{2n} = \prod_{i,j} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{du_i}{2\pi} \frac{dv_j}{2\pi} \left| \prod_{i<j} 2 \sin \left( \frac{u_i - u_j}{2} \right) \frac{2}{\prod_{i<j} 2 \sin \left( \frac{u_i - u_j}{2} \right)} \frac{2}{\prod_{i<j} 2 \sin \left( \frac{v_i - v_j}{2} \right)} \right|^{1/K}$$

Partition function of the impurity contains correlation functions taken at the same point and at different times. Moments of interference experiments come from correlations functions taken at the same time but in different points. Euclidean invariance ensures that the two are the same.
Relation between quantum impurity problem and interference of fluctuating condensates

Normalized amplitude of interference fringes

\[ a^2 = \langle |A_{fr}|^2 \rangle / \langle |A_{fr}|^2 \rangle \]

Distribution function of fringe amplitudes

\[ W(K, a^2) \]

Relation to the impurity partition function

\[ Z_{imp}(K, g) = \int_0^\infty da^2 W(K, a^2) I_0(2g a) \]

Distribution function can be reconstructed from \( Z_{imp}(K, g) \) using completeness relations for the Bessel functions

\[ W(K, a^2) = 2 \int_0^\infty g \, dg \, Z_{imp}(K, ig) J_0(2ga^2) \]
Bethe ansatz solution for a quantum impurity

\( Z_{\text{imp}}(K, g) \) can be obtained from the Bethe ansatz following
Making analytic continuation is possible but cumbersome

Interference amplitude and spectral determinant

\( Z_{\text{imp}}(K, ig) \) is related to the single particle Schroedinger equation


\[
- \frac{d^2 \Psi}{dx^2} + \left( x^{4K-2} + \frac{3}{4x^2} \right) \Psi = E \Psi
\]

Spectral determinant

\[
D(E) = \prod_{n=1}^{\infty} \left( 1 - \frac{E}{E_n} \right)
\]

\[
Z_{\text{imp}}(K, ig) = D \left( \frac{g^2}{\pi^2} (4K)^{2-K^{-1}} \left[ \Gamma(1 - \frac{1}{2K}) \right]^2 \sin^2 \left( \frac{\pi}{2K} \right) \right)
\]
“I think you should be more explicit here in step two.”
Evolution of the distribution function

Narrow distribution for $K \to \infty$. Approaches Gumble distribution.

Width

$$\sigma = \frac{\pi}{6K}$$

Wide Poissonian distribution for $K \to 1$

$$a^2 = |A_{fr}|^2 / \langle |A_{fr}|^2 \rangle$$
From interference amplitudes to conformal field theories

$Z_{\text{imp}}(K, ig)$ correspond to vacuum eigenvalues of $Q$ operators of CFT

When $K>1$, $Z_{\text{imp}}(K, ig)$ is related to $Q$ operators of CFT with $c<0$. This includes 2D quantum gravity, non-intersecting loop model on 2D lattice, growth of random fractal stochastic interface, high energy limit of multicolor QCD, …
Studying coherent dynamics of strongly interacting systems in interference experiments
Coupled 1d systems

Motivated by experiments of Schmiedmayer et al.

\[ \mathcal{H}_0 = \int dx \left[ g n_1^2(x) + \rho (\partial_x \phi_1)^2 \right] + \int dx \left[ g n_2^2(x) + \rho (\partial_x \phi_2)^2 \right] \]

Interactions lead to phase fluctuations within individual condensates

\[ \mathcal{H}_{tun} = -J \int dx \cos(\phi_1 - \phi_2) \]

Tunneling favors aligning of the two phases

Interference experiments measure only the relative phase

\[ \phi_{av} = \frac{\phi_1 + \phi_2}{2} \]

\[ \phi = \phi_1 - \phi_2 \]
Coupled 1d systems

Conjugate variables

\[ \phi = \phi_1 - \phi_2 \quad \Delta n = \frac{(n_1 - n_2)}{2} \]

Relative phase

\[ [\Delta n(x_1), \phi(x_2)] = -i \delta(x_1 - x_2) \]

Particle number imbalance

\[ \mathcal{H}[\phi] = \int dx \, d\tau \left[ \frac{1}{2K}(\Delta n)^2 + \frac{K}{2}(\partial_x \phi)^2 \right] - J \int dx \, d\tau \cos \phi \]

Small \( K \) corresponds to strong quantum fluctuations
Quantum Sine-Gordon model

Hamiltonian

\[ \mathcal{H}[\phi] = \int dx \, d\tau \left[ \frac{1}{2K} (\Delta n)^2 + \frac{K}{2} (\partial_x \phi)^2 \right] - J \int dx \, d\tau \cos \phi \]

Imaginary time action

\[ S[\phi] = \frac{K}{2} \int dx \, d\tau \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + J \int dx \, d\tau \cos \phi \]

Quantum Sine-Gordon model is exactly integrable

Excitations of the quantum Sine-Gordon model

- **soliton** \( \phi = 2\pi \)
- **antisoliton** \( \phi = 0 \)
- **breather**
Coherent dynamics of quantum Sine-Gordon model
Motivated by experiments of Schmiedmayer et al.

Prepare a system at $t=0$

$\phi(x) = 0$

Take to the regime of finite tunneling and let evolve for some time

Measure amplitude of interference pattern
Coherent dynamics of quantum Sine-Gordon model

Oscillations or decay?
From integrability to coherent dynamics

At $t=0$ we have a state with $\phi(x) = 0$ for all $x$

This state can be written as a “squeezed” state

$$\left| \Psi(t = 0) \right\rangle = e^{\sum_{\alpha\beta} \int dp \, K_{\alpha\beta}(p) \, a_{\alpha}^{\dagger}(p) \, a_{\beta}^{\dagger}(-p) \left| \text{vac} \right\rangle$$

Matrix $K_{\alpha\beta}(p)$ can be constructed using connection to boundary SG model
Calabrese, Cardy (2006); Ghoshal, Zamolodchikov (1994)

Time evolution can be easily written

$$\left| \Psi(t) \right\rangle = e^{\sum_{\alpha\beta} \int dp \, e^{-i(E_{\alpha}+E_{\beta})t} \, K_{\alpha\beta}(p) \, a_{\alpha}^{\dagger}(p) \, a_{\beta}^{\dagger}(-p) \left| \text{vac} \right\rangle$$

Interference amplitude can be calculated using form factor approach
Smirnov (1992), Lukyanov (1997)
Coherent dynamics of quantum Sine-Gordon model

Prepare a system at $t=0$

\[ \phi(x) = 0 \]

Take to the regime of finite tunneling and let evolve for some time

Measure amplitude of interference pattern
Coherent dynamics of quantum Sine-Gordon model

Amplitude of interference fringes shows oscillations at frequencies that correspond to energies of breater
Conclusions for part I

Interference of fluctuating condensates can be used to probe correlation functions in one and two dimensional systems. Interference experiments can also be used to study coherent dynamics of interacting systems.
Measuring correlation functions in interacting systems of cold atoms

Lecture II

Quantum noise interferometry in time of flight experiments

1. Time of flight experiments.
   Second order coherence in Mott states of spinless bosons
2. Detection of spin order in Mott states of atomic mixtures
3. Detection of fermion pairing

Emphasis of these lectures:
detection and characterization of many-body quantum states
Bose-Einstein condensation


\[ n \sim 10^{14} \text{cm}^3 \quad T_{\text{BEC}} \sim 1\mu\text{K} \]

Ultralow density condensed matter system

Interactions are weak and can be described theoretically from first principles
Superfluid to Insulator transition

Time of flight experiments

Quantum noise interferometry of atoms in an optical lattice

Second order coherence \( G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle \)
Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment


Hanbury-Brown-Twiss stellar interferometer

\[ \langle I(\vec{r}_1) I(\vec{r}_2) \rangle = A + B \cos \left( \left( \vec{k} - \vec{k}' \right) \cdot (\vec{r}_1 - \vec{r}_2) \right) \]
Hanburry-Brown-Twiss interferometer

\[ E(\vec{r}_1) = E_k e^{i\vec{k}\vec{r}_1} + E_{k'} e^{i\vec{k}'\vec{r}_1} \]

\[ E(\vec{r}_2) = E_k e^{i\vec{k}\vec{r}_2} + E_{k'} e^{i\vec{k}'\vec{r}_2} \]

\[ \langle I(r_1)I(r_2) \rangle = \langle |E(r_1)|^2 |E(r_2)|^2 \rangle \]
\[ = \langle \left( |E_k|^2 + |E_{k'}|^2 + (E_k E_{k'}^* e^{i(\vec{k}-\vec{k}')\vec{r}_1} + \text{c.c.}) \right) \left( |E_k|^2 + |E_{k'}|^2 + (E_k E_{k'}^* e^{i(\vec{k}-\vec{k}')\vec{r}_2} + \text{c.c.}) \right) \rangle \]
\[ = \langle (|E_k|^2 + |E_{k'}|^2)^2 \rangle + \langle |E_k|^2 |E_{k'}|^2 \rangle [e^{i(\vec{k}-\vec{k}')}(\vec{r}_1 - \vec{r}_2) + \text{c.c.}] \]
Second order coherence in the insulating state of bosons

Bosons at quasimomentum \( \vec{k} \) expand as plane waves with wavevectors \( \vec{k}, \vec{k} + \vec{G}_1, \vec{k} + \vec{G}_2 \)

First order coherence: \( \langle \rho(\vec{r}) \rangle \)
Oscillations in density disappear after summing over \( \vec{k} \)

Second order coherence: \( \langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle \)
Correlation function acquires oscillations at reciprocal lattice vectors

\[
\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle = A_0 + A_1 \cos \left( \vec{G}_1 (\vec{r}_1 - \vec{r}_2) \right) + A_2 \cos \left( \vec{G}_2 (\vec{r}_1 - \vec{r}_2) \right) + \ldots
\]
Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment


Effect of parabolic potential on the second order coherence


Width of the correlation peak changes across the transition, reflecting the evolution of Mott domains
The Mott insulator transition in two dimensions

I. B. Spielman,¹,* W. D. Phillips,¹,² and J. V. Porto¹,²

cond-mat/0606216

FIG. 1: Top: expected in situ density profile from a LDA calculation. The extended regions of uniform density denote the portion of the system in the MI phase. Middle: imaged atom density versus final position after TOF. Bottom: noise correlations as a function of final position. Each image represents an average of about 60 raw images and each data set is presented at 3 different values of $t/U$. 

Width of the noise peaks
Interference of an array of independent condensates


Smooth structure is a result of finite experimental resolution (filtering)
Applications of quantum noise interferometry in time of flight experiments

Detection of spin order in Mott states of boson boson mixtures
Engineering magnetic systems using cold atoms in an optical lattice

See also lectures by A. Georges and I. Cirac in this school
Spin interactions using controlled collisions

Two component Bose mixture in optical lattice


$|\uparrow\rangle = |F = 1, m_F = -1\rangle$

$|\downarrow\rangle = |F = 2, m_F = -2\rangle$

Two component Bose Hubbard model

$$\mathcal{H} = -t_{\uparrow} \sum_{\langle ij \rangle} b_{i\uparrow}^\dagger b_{j\uparrow} - t_{\downarrow} \sum_{\langle ij \rangle} b_{i\downarrow}^\dagger b_{j\downarrow} + U_{\uparrow\uparrow} \sum_i n_{i\uparrow}(n_{\uparrow} - 1)$$

$$+ U_{\downarrow\downarrow} \sum_i n_{i\downarrow}(n_{\downarrow} - 1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{\downarrow}$$
Quantum magnetism of bosons in optical lattices

\[ H = J_z \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + J_\perp \sum_{\langle ij \rangle} \left( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) \]

\[ J_z = \frac{t_\uparrow^2 + t_\downarrow^2}{2U_{\uparrow\downarrow}} - \frac{t_\uparrow^2}{U_{\uparrow\uparrow}} - \frac{t_\downarrow^2}{U_{\downarrow\downarrow}} \]

\[ J_\perp = -\frac{t_\uparrow t_\downarrow}{U_{\uparrow\downarrow}} \]

- Ferromagnetic
- Antiferromagnetic

\[ U_{\uparrow\downarrow} \gg U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \]
\[ U_{\uparrow\downarrow} \ll U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \]
Exchange Interactions in Solids

Kinetic energy dominates: antiferromagnetic state

Coulomb energy dominates: ferromagnetic state
Two component Bose mixture in optical lattice. Mean field theory + Quantum fluctuations

Hysteresis

Altman et al., NJP 5:113 (2003)
Probing spin order of bosons

Correlation Function Measurements

\[ G(r_1, r_2) = \langle n(r_1) n(r_2) \rangle_{\text{TOF}} - \langle n(r_1) \rangle_{\text{TOF}} \langle n(r_2) \rangle_{\text{TOF}} \]
\[ \sim \langle n(k_1) n(k_2) \rangle_{\text{LAT}} - \langle n(k_1) \rangle_{\text{LAT}} \langle n(k_2) \rangle_{\text{LAT}} \]

\[ r = \frac{\hbar k t}{m} \]
Engineering exotic phases

- Optical lattice in 2 or 3 dimensions: polarizations & frequencies of standing waves can be different for different directions

- Example: exactly solvable model Kitaev (2002), honeycomb lattice with
  \[
  H = J_x \sum_{\langle i, j \rangle \in x} \sigma_i^x \sigma_j^x + J_y \sum_{\langle i, j \rangle \in y} \sigma_i^y \sigma_j^y + J_z \sum_{\langle i, j \rangle \in z} \sigma_i^z \sigma_j^z
  \]
  - Can be created with 3 sets of standing wave light beams!
  - Non-trivial topological order, “spin liquid” + non-abelian anyons…those has not been seen in controlled experiments
Applications of quantum noise interferometry in time of flight experiments

Detection of fermion pairing
Fermionic atoms in optical lattices

Pairing in systems with repulsive interactions. Unconventional pairing. High Tc mechanism
Fermionic atoms in a three dimensional optical lattice

Kohl et al., PRL 94:80403 (2005)

See also lectures of T. Esslinger and W. Ketterle in this school
Fermions with repulsive interactions

\[ \mathcal{H} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_i n_i \]

Possible d-wave pairing of fermions
High temperature superconductors

Superconducting
Tc 93 K

Hubbard model – minimal model for cuprate superconductors

P.W. Anderson, cond-mat/0201429

After many years of work we still do not understand the fermionic Hubbard model
Positive U Hubbard model


Antiferromagnetic insulator

D-wave superconductor
Second order correlations in the BCS superfluid

\[ \langle \Delta n(r, r')^2 \rangle \]

\[ n(k) \]

\[ k_F \]

\[ n(r') \]

\[ n(r) \]

\[ | \Psi_{BCS} \rangle = \prod_k \left( u_k + v_k c_{k \uparrow}^\dagger c_{-k \downarrow}^\dagger \right) | vac \rangle \]

Expansion of atoms in TOF maps \( k \) into \( r \)

\[ \Delta n(r, r') \equiv n(r) - n(r') \]

\[ \Delta n(r, -r) | \Psi_{BCS} \rangle = 0 \]
Momentum correlations in paired fermions

Greiner et al., PRL 94:110401 (2005)
Fermion pairing in an optical lattice

Second Order Interference
In the TOF images

\[ G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle \]

Normal State

\[ G_N(r_1, r_2) = \delta(r_1 - r_2)\rho(r_1) - \rho^2(r_1) \sum_G \delta(r_1 - r_2 - \frac{G\hbar t}{m}) \]

Superfluid State

\[ G_S(r_1, r_2) = G_N(r_1, r_2) + \Psi(r_1) \sum_G \delta(r_1 + r_2 + \frac{G\hbar t}{m}) \]

\[ \Psi(r) = |u(Q(r))v(Q(r))|^2 \text{ measures the Cooper pair wavefunction} \]

\[ Q(r) = \frac{mr}{\hbar t} \]

One can identify unconventional pairing
Simulation of condensed matter systems: Hubbard Model and high Tc superconductivity

Personal opinion:

The fermionic Hubbard model contains 90% of the physics of cuprates. The remaining 10% may be crucial for getting high Tc superconductivity. Understanding Hubbard model means finding what these missing 10% are. Electron-phonon interaction? Mesoscopic structures (stripes)?

Using cold atoms to go beyond “plain vanilla” Hubbard model
a) Boson-Fermion mixtures: Hubbard model + phonons
b) Inhomogeneous systems, role of disorder
Boson Fermion mixtures

Fermions interacting with phonons
Boson Fermion mixtures

See lectures by T. Esslinger and G. Modugno in this school

Bosons provide cooling for fermions and mediate interactions. They create non-local attraction between fermions.

Charge Density Wave Phase
Periodic arrangement of atoms

Non-local Fermion Pairing
P-wave, D-wave, ...
Boson Fermion mixtures

\[ \mathcal{H} = \mathcal{H}_{bb} + \mathcal{H}_{ff} + \mathcal{H}_{bf} \]
\[ \mathcal{H}_{bb} = -t_b \sum_{\langle ij \rangle} b_i^\dagger b_j + U_{bb} \sum_i n_{bi}(n_{bi} - 1) \]
\[ \mathcal{H}_{ff} = -t_f \sum_{\langle ij \rangle} f_i^\dagger f_j \]
\[ \mathcal{H}_{bf} = U_{bf} \sum_i n_{bi} n_{fi} \]

Effective fermion-"phonon" interaction

\[ \mathcal{\tilde{H}}_{bb} = \sum_q \omega_q \beta_q^\dagger \beta_q \]
\[ \mathcal{\tilde{H}}_{bf} = \sum_{kq} g_q (\beta_q + \beta_{-q}^\dagger) f_{k+q}^\dagger f_k \]

Fermion-"phonon" vertex \( g_q \sim q \)
Similar to electron-phonon systems

"Phonons" : Bogoliubov (phase) mode
Boson Fermion mixtures in 1d optical lattices

Cazalila et al., PRL (2003); Mathey et al., PRL (2004)

\[ \mathcal{H} = -t_b \sum_{\langle ij \rangle} b_i^\dagger b_j - t_f \sum_{\langle ij \rangle} f_i^\dagger f_j - \mu_b \sum_i n_{bi} - \mu_f \sum_i n_{fi} + U_{bb} \sum_i n_{bi}^2 + U_{bf} \sum_i n_{bi} n_{fi} \]

Spinless fermions

Spin \frac{1}{2} fermions
Boson Fermion mixtures in 2d optical lattices

Wang et al., PRA (2005)

40K -- 87Rb

\[ |U_{bd}| / E_r \]

\( \lambda = 1060 \text{nm} \)

40K -- 23Na

\[ |U_{br}| / E_r \]

\( \lambda = 765.5 \text{nm} \)

\( \lambda = 1060 \text{ nm} \)
Conclusions

Interference of extended condensates is a powerful tool for analyzing correlation functions in one and two dimensional systems

Noise interferometry can be used to probe quantum many-body states in optical lattices