Spinor condensates beyond mean-field

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Outline

Introduction

Geometrical classification of spinor condensates

Nematic states of $F=2$ atoms: biaxial, uniaxial, and square.
Energetics: order by disorder

$F=3$ atoms

Spinor condensates in optical lattices
Spin ordering in condensed matter physics
magnetism, triplet Cooper pairing, multicomponent QH systems, …, liquid crystals

Magnetic phase diagram of LiHoF₄

Triplet pairing in $^3$He

Order parameter $\vec{d}(\vec{k})$

Bitko et al., PRL 77:940 (1996)
Spinor condensates in optical traps

Figure courtesy of D. Stamper-Kurn
Spinor condensates in optical traps. S=1 bosons

$$V_{int}(\vec{r_1} - \vec{r_2}) = \left( g_0 + g_2 \vec{F}_1 \cdot \vec{F}_2 \right) \delta(\vec{r_1} - \vec{r_2})$$

Interaction energy

Ferromagnetic condensate for $g_2<0$. Realized for $^{87}$Rb. Favors $F_z = \pm 1$

Antiferromagnetic condensate for $g_2>0$. Realized for $^{23}$Na. Favors $F_z = 0$

$$E_{grad} = - \int d^3 \vec{r} p(z) n \langle F_z \rangle$$

Linear Zeeman

$$E_{quad} = q \int d^3 \vec{r} n \langle F_z^2 \rangle$$

Quadratic Zeeman

Stamper-Kurn, Ketterle, cond-mat/0005001

See also
Ho, PRL 81:742 (1998)
S=1 antiferromagnetic condensate

Stamper-Kurn, Ketterle, cond-mat/0005001

Ground state spin domains in F=1 spinor condensates

Representation of ground-state spin-domain structures. The spin structures correspond to long vertical lines through the spin-domain diagram.
Coherent dynamics of spinor condensates
Ramsey experiments with spin-1 atoms

Kronjager et al., PRA 72:63619 (2005)
Coherent dynamics of spinor condensates


FIG. 3: (a) Spin population oscillations in $f = 1$ between $|1, 0; 1, 0\rangle$ (○) and $|1, +1; 1, -1\rangle$ (●) at a magnetic field of 0.28 G and a lattice

FIG. 4: (a) Measured population dynamics for the coupled three-level system $|2, 0; 2, 0\rangle \leftrightarrow |2, +1; 2, -1\rangle \leftrightarrow |2, +2; 2, -2\rangle$ at a mag-
Classification of spinor condensates
How to classify spinor states

Traditional classification is in terms of order parameters

Spin $\frac{1}{2}$ atoms (two component Bose mixture)

Spin 1 atoms

\[
\langle Q_{ab} \rangle = \frac{1}{2} \langle F_a F_b + F_b F_a \rangle - \frac{1}{3} F^2
\]

Nematic order parameter. Needed to characterize e.g. $F=1, F_z=0$ state

This approach becomes very cumbersome for higher spins
Classification of spinor condensates

How to recognize fundamentally distinct spinor states?

States of F=2 bosons. All equivalent by rotations.

Introduce “Spin Nodes”

$|\hat{n}\rangle$ -- coherent state

$\langle \hat{n} | \psi \rangle = 0$

2F maximally polarized states orthogonal to $|\psi\rangle$

4F degrees of freedom

Classification of spinor condensates

Introduce fully polarized state in the direction \( \hat{n}(\theta, \phi) \)

Stereographic mapping into the complex plane

\[
\zeta = e^{i\phi} \tan \theta/2
\]

\[
F \cdot \hat{n} |\zeta\rangle = F |\zeta\rangle \\
|\zeta\rangle = \sum_{\alpha=0}^{2F} \sqrt{\binom{2F}{\alpha}} \zeta^\alpha |F - \alpha\rangle,
\]

Characteristic polynomial for a state

\[
|\psi\rangle = \sum_{\alpha=-F}^{\alpha=F} A_{\alpha} |\alpha\rangle
\]

\[
f_\psi(\zeta) \equiv \langle \psi | \zeta \rangle = \sum_{\alpha=0}^{2F} \sqrt{\binom{2F}{\alpha}} A_{F-\alpha}^* \zeta^\alpha
\]

2F complex roots \( \{(\theta_i, \phi_i)\} \) of \( f_\psi(\zeta) = 0 \) determine \( |\psi\rangle \)

Symmetries of \( |\psi\rangle \) correspond to symmetries of the set of points \( \{(\theta_i, \phi_i)\} \)
Classification of spinor condensates. F=1

Ferromagnetic states

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

Orthogonal state

\[ |m_S = -1 \rangle \]

Two degenerate “nodes” at the South pole

Polar (nematic) state

\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

Orthogonal states

\[ |m_S = 1 \rangle \]
\[ |m_S = -1 \rangle \]
Classification of spinor condensates. F=2

\[ V_{\text{int}}(x_1 - x_2) = \delta(x_1 - x_2)(g_0 P_0 + g_2 P_2 + g_4 P_4) \]

\[ A_\alpha = \left( \frac{\sin(\eta)}{\sqrt{2}}, 0, \cos(\eta), 0, \frac{\sin(\eta)}{\sqrt{2}} \right) \]

Nematic

\[ a_2^+ - a_4 \]

Ferromagnetic

\[ A_\alpha = (1, 0, 0, 0, 0) \]

\[ a_0^- - a_4 \]

Tetrahedratic

\[ A_\alpha = \left( \sqrt{\frac{1}{3}}, 0, 0, \sqrt{\frac{2}{3}}, 0 \right) \]

\[ ^{23}\text{Na} \]

\[ ^{87}\text{Rb} \]

Ciobanu, Yip, Ho,
Classification of spinor condensates. $F=3$

(a) \( \left( \frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}} \right) \)

(b) \( (\sin(\eta), 0, 0, 0, \cos(\eta), 0) \)

(c) \( \left( \frac{\sin(\eta)}{\sqrt{2}}, 0, 0, \cos(\eta), 0, \frac{\sin(\eta)}{\sqrt{2}} \right) \)

(d) \( \left( 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0 \right) \)
Mathematics of spinor classification

Classified polynomials in two complex variables according to tetrahedral, icosohedral, etc. symmetries

Felix Klein

\[ z_1^k z_2^{n-k} \quad \rightarrow \quad |S = \frac{n}{2}; m_S = \frac{n}{2} - k \rangle \]
Novel states of spinor condensates:
uniaxial, biaxial, and square nematic states for $S=2$
F=2 spinor condensates

\[ V_{\text{int}}(x_1 - x_2) = \delta(x_1 - x_2)(g_0 \mathcal{P}_0 + g_2 \mathcal{P}_2 + g_4 \mathcal{P}_4) \]

But… unusual degeneracy of the nematic states
Nematic states of F=2 spinor condensates

Degeneracy of nematic states at the mean-field level

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \sin \eta/\sqrt{2} \\ 0 \\ \cos \eta \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

Square nematic

Biaxial nematic

Uniaxial nematic

Nematic states of F=2 spinor condensates

Uniaxial

Two spin wave excitations

One vortex (no multiplicity of the phase winding)

Biaxial

Three spin wave excitations

Three types of vortices with spin twisting

$4 \times Z_v$ different vortices

One type of vortices without spin twisting

Square

Three spin wave excitations

Five types of vortices with spin twisting

$6 \times Z_v$ different vortices

One type of vortices without spin twisting

Non-Abelian fundamental group
Spin twisting vortices in biaxial nematics
Mermin, Rev. Mod. Phys. 51:591 (1979)

Disclination in both sticks
Disclination in long stick
Disclination in short stick
Spin textures in liquid crystal nematics

Defect morphology in a biaxial thermotropic polymer

T. De’Neve (1), M. Kleman (2) and P. Navard (3)  J. Phys. II France 2 (1992) 187-207

Abstract. — Optical observations of the thread texture of a nematic thermotropic polymer commercially known as VECTRA B950® have shown the existence of three types of half integer disclination lines, called here $E_x$, $E_y$ and $E_z$, sometimes associated with integer disclination lines. This plurality of defects is typical of the biaxial nature of this nematic phase, due to interchain rotational correlations between phenyl rings. The relationships between defects which arise from the topological theory are confirmed, in static as well as in dynamic experiments. During shear
How the nematics decide. Bi-, Uni-, or Square-?
Nematic states of F=2 condensates. Order by disorder at T=0

Energy of zero point fluctuations \( k_B T \ll m v_i^2 \)

\[
E_{\text{zero-point}} = \frac{1}{2} \sum_{i=1}^{5} \int \hbar \omega_i(k) \frac{d^3k}{(2\pi)^3}
\]

The frequencies of the modes depend on the ground state spinor

Uniaxial nematic with fluctuation

\[
\begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} + \epsilon \begin{pmatrix}
0 \\
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]

\( \langle \vec{F} \rangle^2 \) bigger

Square nematic with fluctuation

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
1
\end{pmatrix} + \epsilon \begin{pmatrix}
0 \\
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]

\( \langle \vec{F} \rangle^2 \) smaller
Nematic states of F=2 condensates.
Order by disorder at T=0

\[ \frac{\Delta E}{N} \approx 0.3 \text{ pK} \] for Rb

\[ E_{\text{quad}} \approx -1.6 \text{ nK} \times F_z^2 \]
at magnetic field B=340 mG
Nematic states of F=2 condensates.
Order by disorder at finite temperature

Thermal fluctuations further separate uniaxial and square nematic condensates

Effect of the magnetic field

Overcomes 30mG

FIG. 3: Evolution of the uniaxial nematic into square biaxial-nematic in a field. The free-energy, with a constant subtracted, is plotted in pK per atom for the conditions of Ref [2], for $B = 0, 20, 27$mG (top to bottom). Reciprocal spinors are also shown. At the minima, they deform continuously from rectangles into squares as the field is increased. The Ising-like symmetry breaking at the critical field corresponds to making a choice between which way to deform the square.
Quantum phase transition in the nematic state of F=2 atoms

Will the spins thermolize as we cross phase boundaries?
Generation of topological defects in nematic liquid crystals by crossing phase transition lines

Defect tangle after a temperature quench

Chuang et al., Science 251:1336 (1991)

Coarsening dynamics of defects after the pressure quench

Chuang et al., PRA 47:3343 (1993)
$F=3$ spinor condensates

Motivated by BEC of $^{52}\text{Cr}$: Griesmaier et al., PRL 94:160401
F=3 spinor condensates

Barnett, Turner, Demler, cond-mat/0611230

See also
Santos, Pfau,
PRL 96:190404 (2006);
Diener, Ho,
F=3 spinor condensates. Vortices

|     | $A_\alpha$                                                                 | $|\langle S \rangle|$ | $SO(3)$ subgroup | $N_{vort}$ |
|-----|---------------------------------------------------------------------------|---------------------|-----------------|-----------|
| A   | $(\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}})$                  | 0                   | $D_6$           | $9 \times \mathbb{Z}_v$ |
| B   | $(0, \frac{\sin(\eta)}{\sqrt{2}}, 0, \cos(\eta), 0, -\frac{\sin(\eta)}{\sqrt{2}}, 0)$ | 0                   | $D_2$           | $5 \times \mathbb{Z}_v$ |
| C   | $(0, \frac{\sin(\chi) \cos(\eta)}{\sqrt{2}}, \frac{i \sin(\chi) \sin(\eta)}{\sqrt{2}}, \cos(\chi), -\frac{i \sin(\chi) \sin(\eta)}{\sqrt{2}}, \frac{\sin(\chi) \cos(\eta)}{\sqrt{2}}, 0)$ | 0                   | $C_2$           | $4 \times \mathbb{Z}_v$ |
| D   | $(0, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0)$                | 0                   | $O$             | $8 \times \mathbb{Z}_v$ |
| E   | $(\frac{\sin(\eta)}{\sqrt{2}}, 0, 0, \cos(\eta), 0, 0, \frac{\sin(\eta)}{\sqrt{2}})$ | 0                   | $C_3$           | $6 \times \mathbb{Z}_v$ |
| F   | $(0, 1, 0, 0, 0, 0, 0)$                                                   | 2                   | $U(1)$          | 4         |
| FF  | $(1, 0, 0, 0, 0, 0, 0)$                                                   | 3                   | $U(1)$          | 6         |
| G   | $(0, \sin(\eta), 0, \cos(\eta), 0, 0, 0)$                                | $2 \sin(\eta)$     | $C_2$           | $4 \times \mathbb{Z}_v$ |
| H   | $(\sin(\eta), 0, 0, 0, 0, \cos(\eta), 0)$                                | $|3 \sin(\eta) - 2 \cos(\eta)|$ | $C_5$           | $10 \times \mathbb{Z}_v$ |
| HH  | $(0, \sin(\eta), 0, 0, \cos(\eta), 0, 0)$                                | $|2 \sin(\eta) - \cos(\eta)|$ | $C_3$           | $6 \times \mathbb{Z}_v$ |
Enhancing spin interactions

Two component bosons in an optical lattice
Superfluid to insulator transition in an optical lattice

Two component Bose mixture in optical lattice


$|\uparrow\rangle = |F = 1, m_F = -1\rangle$

$|\downarrow\rangle = |F = 2, m_F = -2\rangle$

Two component Bose Hubbard model

\[
\mathcal{H} = - t_\uparrow \sum_{\langle ij \rangle} b_{i\uparrow}^\dagger b_{j\uparrow} - t_\downarrow \sum_{\langle ij \rangle} b_{i\downarrow}^\dagger b_{j\downarrow} + U_{\uparrow\uparrow} \sum_i n_{i\uparrow} (n_{\uparrow} - 1)
\]

\[
+ U_{\downarrow\downarrow} \sum_i n_{i\downarrow} (n_{\downarrow} - 1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{\downarrow}
\]
Two component Bose mixture in optical lattice. Magnetic order in an insulating phase

Insulating phases with N=1 atom per site. Average densities $n_\uparrow = n_\downarrow = \frac{1}{2}$

Easy plane ferromagnet

$$| \Psi \rangle = \prod_i \left( b_{i\uparrow}^\dagger + e^{i\phi} b_{i\downarrow}^\dagger \right) | 0 \rangle$$

Easy axis antiferromagnet

$$| \Psi \rangle = \prod_{i \in A} b_{i\uparrow}^\dagger \prod_{i \in B} b_{i\downarrow}^\dagger$$
Quantum magnetism of bosons in optical lattices

Duan, Lukin, Demler, PRL (2003)

\[ \mathcal{H} = J_z \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + J_\perp \sum_{\langle ij \rangle} \left( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) \]

\[ J_z = \frac{t_\uparrow^2 + t_\downarrow^2}{2U_{\uparrow\downarrow}} - \frac{t_\uparrow^2}{U_{\uparrow\uparrow}} - \frac{t_\downarrow^2}{U_{\downarrow\downarrow}} \]

\[ J_\perp = -\frac{t_\uparrow t_\downarrow}{U_{\uparrow\downarrow}} \]

- Ferromagnetic
- Antiferromagnetic

\[ U_{\uparrow\downarrow} \gg U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \]

\[ U_{\uparrow\downarrow} \ll U_{\uparrow\uparrow}, U_{\downarrow\downarrow} \]
How to detect antiferromagnetic order?

Quantum noise measurements in time of flight experiments
Time of flight experiments

Quantum noise interferometry of atoms in an optical lattice

Second order coherence

\[ G(r_1, r_2) = \langle n(r_1)n(r_2) \rangle - \langle n(r_1) \rangle \langle n(r_2) \rangle \]
Second order coherence in the insulating state of bosons.
Hanburry-Brown-Twiss experiment

Hanbury-Brown-Twiss stellar interferometer

\[ \langle I(\vec{r}_1) I(\vec{r}_2) \rangle = A + B \cos \left( (\vec{k} - \vec{k}') (\vec{r}_1 - \vec{r}_2) \right) \]
Second order coherence in the insulating state of bosons

Bosons at quasimomentum $\vec{k}$ expand as plane waves with wavevectors $\vec{k}$, $\vec{k} + \vec{G}_1$, $\vec{k} + \vec{G}_2$

First order coherence: $\langle \rho(\vec{r}) \rangle$
Oscillations in density disappear after summing over $\vec{k}$

Second order coherence: $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle$
Correlation function acquires oscillations at reciprocal lattice vectors

$$\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle = A_0 + A_1 \cos \left( \vec{G}_1(\vec{r}_1 - \vec{r}_2) \right) + A_2 \cos \left( \vec{G}_2(\vec{r}_1 - \vec{r}_2) \right) + \ldots$$
Second order coherence in the insulating state of bosons. Hanbury-Brown-Twiss experiment


Second order coherence in the insulating state of fermions. Hanburry-Brown-Twiss experiment


Experiment: T. Tom et al. Nature in press
Probing spin order of bosons

Extra Bragg peaks appear in the second order correlation function in the AF phase

\[ G(r_1, r_2) = \langle n(r_1) n(r_2) \rangle_{\text{TOF}} - \langle n(r_1) \rangle_{\text{TOF}} \langle n(r_2) \rangle_{\text{TOF}} \]

\[ \approx \langle n(k_1) n(k_2) \rangle_{\text{LAT}} - \langle n(k_1) \rangle_{\text{LAT}} \langle n(k_2) \rangle_{\text{LAT}} \]
Realization of spin liquid using cold atoms in an optical lattice

Theory: Duan, Demler, Lukin PRL 91:94514 (03)

Kitaev model

\[ H = - J_x \sum \sigma_i^x \sigma_j^x - J_y \sum \sigma_i^y \sigma_j^y - J_z \sum \sigma_i^z \sigma_j^z \]

Questions:
Detection of topological order
Creation and manipulation of spin liquid states
Detection of fractionalization, Abelian and non-Abelian anyons
Melting spin liquids. Nature of the superfluid state
Enhancing the role of interactions. \( F=1 \) atoms in an optical lattice
Antiferromagnetic spin F=1 atoms in optical lattices

$$V_{\text{int}}(\vec{r}_1 - \vec{r}_2) = \left(g_0 + g_2 \vec{F}_1 \cdot \vec{F}_2\right) \delta(\vec{r}_1 - \vec{r}_2)$$

Hubbard Hamiltonian

$$\mathcal{H} = -t \sum_{\langle ij \rangle} a_{i}^\dagger a_{j} + U_0 \sum_i n_i^2 + U_2 \sum_i \vec{S}_i^2 - \mu \sum_i n_i$$

Symmetry constraints

$$n_i + S_i = \text{even}$$

Antiferromagnetic spin $F=1$ atoms in optical lattices

**Hubbard Hamiltonian**  

$$\mathcal{H} = -t \sum_{\langle ij \rangle} a_{i m}^\dagger a_{j m} + U_0 \sum_i n_i^2 + U_2 \sum_i \vec{S}_i^2 - \mu \sum_i n_i$$

**Symmetry constraints**

$$n_i + S_i = \text{even}$$

---

**Nematic Mott Insulator**

$$|\Psi\rangle = \prod_i (n_x a_{ix}^\dagger + n_y a_{iy}^\dagger + n_z a_{iz}^\dagger)^N |0\rangle$$

**Spin Singlet Mott Insulator**

$$|\Psi\rangle = \prod_i (a_{ix}^\dagger a_{iy}^\dagger a_{iz}^\dagger)^{N/2} |0\rangle$$

Law et al., PRL 81:5257 (1998)  
Ho, Yip, PRL 84:4031 (2000)
Nematic insulating phase for N=1

Effective S=1 spin model \[ \mathcal{H} = -J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - J_2 \sum_{\langle ij \rangle} \left( \vec{S}_i \cdot \vec{S}_j \right)^2 \]

\[ J_1 = \frac{2t^2}{U_0 + U_2} \]
\[ J_2 = \frac{2t^2}{3(U_0 + U_2)} + \frac{4t^2}{3(U_0 - U_2)} \]

When \[ J_2 > J_1 \] the ground state is nematic in d=2,3.

\[ \langle S_a \rangle = 0 \]
\[ \langle S_a S_b \rangle \neq 0 \]

One dimensional systems are dimerized: see e.g. Rizzi et al., PRL 95:240404 (2005)
Nematic insulating phase for N=1.

Two site problem

\[ \mathcal{H} = -J_1 \sum_{\langle ij \rangle} \vec{S}_i \vec{S}_j - J_2 \sum_{\langle ij \rangle} \left( \vec{S}_i \vec{S}_j \right)^2 \]

<table>
<thead>
<tr>
<th>( S_{\text{tot}} )</th>
<th>( \vec{S}_1 \vec{S}_2 )</th>
<th>( \left( \vec{S}_1 \vec{S}_2 \right)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
<td>4</td>
</tr>
</tbody>
</table>

Singlet state is favored when \( J_2 > J_1 \).

One can not have singlets on neighboring bonds. Nematic state is a compromise. It corresponds to a superposition of \( S_{\text{tot}} = 0 \) and \( S_{\text{tot}} = 2 \) on each bond.
Conclusions

Spinor condensates can be represented as polyhedra. Symmetries of spinor states correspond to rotation symmetries of polyhedra.

F=2 condensates. Mean-field degeneracy of nematic states: uniaxial, biaxial, square. Degeneracy lifted by fluctuations. Transition between biaxial and square nematics in a magnetic field.


Spinor condensates in an optical lattice. Exchange interactions in the insulating states can lead to various kinds of magnetic ordering.