Interference of fluctuating condensates

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Outline

Measuring correlation functions in interference experiments

1. Interference of independent condensates
2. Interference of interacting 1D systems
3. Interference of 2D systems
4. Full distribution function of fringe visibility in interference experiments. Connection to quantum impurity problem

Studying decoherence in interference experiments.
Effects of finite temperature

Distribution function of magnetization for lattice spin systems
Measuring correlation functions in interference experiments
Interference of two independent condensates

Interference of two independent condensates

Clouds 1 and 2 do not have a well defined phase difference. However each individual measurement shows an interference pattern.

\[
\rho_{\text{int}}(r) = a_1^\dagger(r) a_2(r) + \text{c.c.}
\]

\[
a_1(r) = e^{i \phi_1 + i k_1 r}
\]

\[
k_1 = \frac{m v_1}{\hbar} = \frac{m r}{\hbar t}
\]

\[
a_2(r) = e^{i \phi_2 + i k_2 r}
\]

\[
k_2 = \frac{m v_2}{\hbar} = \frac{m (r + d)}{\hbar t}
\]

\[
\rho_{\text{int}}(r) = e^{i (k_2 - k_1) r} e^{i (\phi_2 - \phi_1)} + \text{c.c.}
\]

\[
\rho_{\text{int}}(r) = e^{i \frac{m d r}{\hbar t}} e^{i (\phi_2 - \phi_1)} + \text{c.c.}
\]

\[
\langle \rho_{\text{int}}(r) \rangle = 0
\]

\[
\langle \rho_{\text{int}}(r) \rho_{\text{int}}(r') \rangle = e^{i \frac{m d}{\hbar t} (r - r')} + \text{c.c.}
\]
Interference of one dimensional condensates


Reduction of the contrast due to fluctuations

Figures courtesy of J. Schmiedmayer
Interference of one dimensional condensates

Amplitude of interference fringes, $A_{fr}$, contains information about phase fluctuations within individual condensates.

$$d\rho_{int}(x, y) = (e^{i \frac{mdy}{\hbar t}} a_1^\dagger(x) a_2(x) + \text{c.c.}) \, dx$$

$$\rho_{int}(y) = e^{i \frac{mdy}{\hbar t}} \int_{0}^{L} dx \, a_1^\dagger(x) a_2(x) + \text{c.c.}$$

$$\rho_{int}(y) = A_{fr} e^{i \Delta \phi + i \frac{mdy}{\hbar t}} + \text{c.c.}$$
Interference amplitude and correlations

Polkovnikov, Altman, Demler, PNAS 103:6125 (2006)

\[ A_{fr} e^{i \Delta \phi} = \int_0^L dx \ a_1^\dagger(x) a_2(x) \]

\[ \langle |A_{fr}|^2 \rangle = \int_0^L \int_0^L dx \, dy \ \langle a_1^\dagger(x) a_2(x) a_2^\dagger(y) a_1(y) \rangle \]

\[ \simeq L \int_0^L dx \ \langle a_1(x) a_1^\dagger(0) \rangle \langle a_2(0) a_2^\dagger(x) \rangle \]

For identical condensates

\[ \langle |A_{fr}|^2 \rangle = L \int_0^L dx \ (G(x))^2 \]

Instantaneous correlation function

\[ G(x) = \langle a(x) a^\dagger(0) \rangle \]
Interference between Luttinger liquids

Luttinger liquid at $T=0$

$$G(x) \sim \rho \left( \frac{\xi_h}{x} \right)^{1/2K}$$

$K$ – Luttinger parameter

$$\langle |A_{fr}|^2 \rangle \sim \left( \rho \xi_h \right)^{1/K} \left( L \rho \right)^{2-1/K}$$

For non-interacting bosons $K = \infty$ and $A_{fr} \sim L$

For impenetrable bosons $K = 1$ and $A_{fr} \sim \sqrt{L}$

Luttinger liquid at finite temperature

$$\langle |A_{fr}|^2 \rangle \sim L \rho^2 \xi_h \left( \frac{\hbar^2}{m \xi_h^2} \frac{1}{T} \right)^{1-1/K}$$

Analysis of $A_{fr}$ can be used for thermometry
Rotated probe beam experiment

\[ \langle |A_{fr}|^2 \rangle = L \int_0^L dx \cos(qx)(G(x))^2 \]

For large imaging angle, \( qL \gg 1 \)

\[ \langle |A_{fr}|^2 \rangle \sim L \rho^2 \xi_h \sin\left(\frac{\pi}{K}\right) \Gamma\left(1 - \frac{2}{K}\right) (\xi_h q)^{1/K-1} \]

Luttinger parameter \( K \) may be extracted from the angular dependence of \( A_{fr}(\theta) \)

\[ q = \frac{md}{\hbar t \tan \theta} \]
Interference between two-dimensional BECs at finite temperature. Kosteritz-Thouless transition
Interference of two dimensional condensates


Probe beam parallel to the plane of the condensates

\[
\langle |A_{fr}|^2 \rangle = L_x L_y \int_0^{L_x} \int_0^{L_y} d^2\vec{r} \left( G(\vec{r}) \right)^2
\]

\[
G(\vec{r}) = \langle a(\vec{r}) a^\dagger(0) \rangle
\]
Interference of two dimensional condensates. Quasi long range order and the KT transition


Above KT transition

\[ G(r) \sim e^{-r/\xi} \]

\[ \langle |A_{fr}|^2 \rangle \sim L_x L_y \]

\[ \log \xi(T) \sim 1/\sqrt{T - T_{KT}} \]

Below KT transition

\[ G(r) \sim \rho \left( \frac{\xi_h}{r} \right)^\alpha \]

\[ \alpha(T) = \frac{mT}{2\pi \rho_s(T) \hbar^2} \]

\[ \langle |A_{fr}|^2 \rangle \sim (L_x L_y)^{2-\alpha} \]
Experiments with 2D Bose gas


Typical interference patterns

low temperature

higher temperature
Experiments with 2D Bose gas

Hadzibabic et al., Nature 441:1118 (2006)
Experiments with 2D Bose gas

Hadzibabic et al., Nature 441:1118 (2006)

fit by:

\[ C^2 \sim \frac{1}{D_x} \int_0^{D_x} [g_1(0, x)]^2 dx \sim \left( \frac{1}{D_x} \right)^{2\alpha} \]

**Exponent \( \alpha \)**

- if \( g_1(r) \) decays exponentially with \( \ell_{\text{coh}} \ll D_x \): \( \alpha = 1/2 \)
- if \( g_1(r) \) decays algebraically or exponentially with a large \( \ell_{\text{coh}} \): \( \alpha < 1/2 \)

**“Sudden” jump!?**
Experiments with 2D Bose gas

Hadzibabic et al., Nature 441:1118 (2006)

c.f. Bishop and Reppy

He experiments:
universal jump in the superfluid density

Ultracold atoms experiments:
jump in the correlation function.
KT theory predicts $\alpha=1/4$
just below the transition
Experiments with 2D Bose gas. Proliferation of thermal vortices

Hadzibabic et al., Nature 441:1118 (2006)

Fraction of images showing at least one dislocation

The onset of proliferation coincides with $\alpha$ shifting to 0.5!
Full distribution function of fringe amplitudes for interference experiments between two 1d condensates

Higher moments of interference amplitude

$A_{fr}$ is a quantum operator. The measured value of $|A_{fr}|$ will fluctuate from shot to shot. Can we predict the distribution function of $|A_{fr}|$?

Higher moments

$$
\langle |A_{fr}|^{2n} \rangle = \int_0^L dz_1 \ldots dz_n \langle a^\dagger(z_1) \ldots a^\dagger(z_n) a(z'_1) \ldots a(z'_n) \rangle^2
$$

Changing to periodic boundary conditions (long condensates)

$$
\langle |A_{fr}|^{2n} \rangle = \langle |A_{fr}|^2 \rangle^n \times Z_{2n}
$$

$$
Z_{2n} = \prod_{i,j} \int_0^{2\pi} \frac{du_i}{2\pi} \int_0^{2\pi} \frac{dv_j}{2\pi} \left| \frac{\prod_{i<j} 2 \sin(\frac{u_i-u_j}{2}) \prod_{i<j} 2 \sin(\frac{v_i-v_j}{2})}{\prod_{i,j} 2 \sin(\frac{u_i-v_j}{2})} \right|^{1/K}
$$

Explicit expressions for $Z_{2n}$ are available but cumbersome

Impurity in a Luttinger liquid

\[ S = \frac{\pi K}{2} \int dx \, d\tau \left[ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] + 2g \int d\tau \cos \phi(x = 0, \tau) \]

Expansion of the partition function in powers of \( g \)

\[ Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(2n)!} \int d\tau_1 \ldots d\tau_n \left( e^{i\phi} + e^{-i\phi} \right)_{\tau_1} \ldots \left( e^{i\phi} + e^{-i\phi} \right)_{\tau_{2n}} \]
\[ Z_{\text{imp}} = \sum_n \frac{g^{2n}}{(n!)^2} Z_{2n} \]
\[ Z_{2n} = \left| \prod_{i,j} \int_0^{2\pi} du_i \, dv_j \int_0^{2\pi} du_i \, dv_j \left[ \prod_{i<j} 2 \sin \left( \frac{u_i - u_j}{2} \right) \prod_{i<j} 2 \sin \left( \frac{v_i - v_j}{2} \right) \right] \right|^{1/K} \]

Partition function of the impurity contains correlation functions taken at the same point and at different times. Moments of interference experiments come from correlations functions taken at the same time but in different points. Euclidean invariance ensures that the two are the same.
Relation between quantum impurity problem and interference of fluctuating condensates

Normalized amplitude of interference fringes

\[ a^2 = |A_{fr}|^2 / \langle |A_{fr}|^2 \rangle \]

Distribution function of fringe amplitudes

\[ W( K, a^2 ) \]

Relation to the impurity partition function

\[ Z_{imp}( K, g ) = \int_0^\infty da^2 W( K, a^2 ) I_0( 2g a ) \]

Distribution function can be reconstructed from using completeness relations for the Bessel functions

\[ W(K, a^2) = 2 \int_0^\infty g\, dg\, Z_{imp}(K, ig) J_0(2ga^2) \]
Bethe ansatz solution for a quantum impurity

$Z_{\text{imp}}(K, g)$ can be obtained from the Bethe ansatz following
Making analytic continuation is possible but cumbersome

Interference amplitude and spectral determinant

$Z_{\text{imp}}(K, ig)$ is related to the Schroedinger equation


$-\frac{d^2}{dx^2} \Psi + \left( x^{4K-2} + \frac{3}{4x^2} \right) \Psi = E \Psi$

Spectral determinant $D(E) = \prod_{n=1}^{\infty} \left( 1 - \frac{E}{E_n} \right)$

$Z_{\text{imp}}(K, ig) = D \left( \frac{g^2}{\pi^2} (4K)^{2-K^{-1}} \left[ \Gamma(1 - \frac{1}{2K}) \right]^2 \sin^2\left( \frac{\pi}{2K} \right) \right)$
"I think you should be more explicit here in step two."
Evolution of the distribution function

\[ a^2 = |A_{fr}|^2 / \langle |A_{fr}|^2 \rangle \]

Narrow distribution for \( K \to \infty \)
Approaches Gumbel distribution.

Width \( \sigma = \frac{\pi}{6K} \)

Wide Poissonian distribution for \( K \to 1 \)
Gumbel distribution and 1/f noise

For $K \gg 1$

Fringe visibility

\[ \text{Roughness} = \int h(\varphi)^2 \, d\varphi \]

1/f Noise and Extreme Value Statistics
Gumbel Distribution in Statistics

Describes Extreme Value Statistics, appears in climate studies, finance, etc.

Stock performance: distribution of “best performers” for random sets chosen from S&P500

Distribution of largest monthly rainfall over a period of 291 years at Kew Gardens
Distribution function for open boundary conditions, finite temperature, 2D systems, ...

A. Imambekov et al.

\[ Z_{2n} = \int_{0}^{1} \ldots \int_{0}^{1} du_{1} \ldots du_{n} dv_{1} \ldots dv_{n} \left( \frac{\prod_{i<j} |u_{i} - u_{j}| \prod_{i<j} |v_{i} - v_{j}|}{\prod_{i,j} |u_{i} - v_{j}|} \right)^{1/K} \]

\[ = \int_{-1/2}^{1/2} \ldots \int_{-1/2}^{1/2} du_{1} \ldots dv_{n} e^{\frac{1}{K} \left( \sum_{i<j} f(u_{i}, u_{j}) + \sum_{i<j} f(v_{i}, v_{j}) - \sum_{i,j} f(u_{i}, v_{j}) \right)} \]

\[ f^{p}(x, y) = \log \frac{1}{\pi} \sinh |\pi(x - y)| \]

Periodic boundary conditions

\[ f(x, y) = \log (|x - y|) \]

Open boundary conditions

\[ f(x, y, a) = \log \frac{a}{\pi} \sinh |\frac{\pi(x - y)}{a}|, a = \frac{\xi_{T}}{L} \]

Finite temperature

Partition function of classical plasma

1. High temperature expansion (expansion in 1/K)

2. Non-perturbative solution
Periodic versus Open boundary conditions

\[ a^2 = \frac{|A_{fr}|^2}{\langle |A_{fr}|^2 \rangle} \]

K=5

periodic
open
Distribution functions at finite temperature

\[ \xi_T \]

\[ \frac{\xi_T}{L} = \infty \]

K=5

\[ \frac{\xi_T}{L} = 0.2 \]

\[ \frac{\xi_T}{L} = 0.05 \]

\[ \langle a^+(z)a(0) \rangle \sim \rho \xi_h^{1/2K} \left( \frac{\pi/\xi_T}{\sinh(\pi z/\xi_T)} \right)^{1/2K} \]

For K>>1, crossover at \( \xi_T K \sim L \)

\[ a^2 = |A_{fr}|^2 / \langle |A_{fr}|^2 \rangle \]
Interference between fluctuating 2d condensates. Distribution function of the interference amplitude

\[ a^2 = \frac{|A_{fr}|^2}{\langle |A_{fr}|^2 \rangle} \]

K=5
K=3
K=2, at BKT transition

A. Imambekov et al. preprint
aspect ratio 1 (Lx=Ly).
When $K>1$, $Z_{\text{imp}}(K, ig)$ is related to $Q$ operators of CFT with $c<0$. This includes 2D quantum gravity, non-intersecting loop model on 2D lattice, growth of random fractal stochastic interface, high energy limit of multicolor QCD, …

$Z_{\text{imp}}(K, ig)$ correspond to vacuum eigenvalues of $Q$ operators of CFT. 

From interference amplitudes to conformal field theories

![Graph showing the relationship between $C$ and $K$. Points indicate 2D quantum gravity, non-intersecting loops on 2D lattice, and Yang-Lee singularity.]
Condensate decoherence at finite temperature probed with interference experiments
Studying dynamics using interference experiments.
Thermal decoherence

Prepare a system by splitting one condensate
Take to the regime of zero tunneling
Measure time evolution of fringe amplitudes
Finite temperature phase dynamics

\[ \mathcal{H}_0 = \int dx \left[ g n_1^2(x) + \rho (\partial_x \phi_1)^2 \right] + \int dx \left[ g n_2^2(x) + \rho (\partial_x \phi_2)^2 \right] \]

Temperature leads to phase fluctuations within individual condensates

Interference experiments measure only the relative phase

\[ \phi_{av} = \frac{\phi_1 + \phi_2}{2} \]

\[ \phi = \phi_1 - \phi_2 \]
Relative phase dynamics

\[ \mathcal{H} = \int d^d r \left[ \frac{g}{2} (\Delta n)^2 + \frac{\rho}{2} (\nabla \phi)^2 \right] \]

Hamiltonian can be diagonalized in momentum space

A collection of harmonic oscillators with

\[ \omega_q = \sqrt{g\rho} |q| \]

Need to solve dynamics of harmonic oscillators at finite T

Conjugate variables

\[ \phi = \phi_1 - \phi_2 \]

\[ \Delta n = (n_1 - n_2) / 2 \]

Coherence

\[ \langle \Psi(t) | e^{i\phi} | \Psi(t) \rangle = e^{-\frac{1}{2} \sum_q \langle \phi_q^2(t) \rangle} \]
Relative phase dynamics

High energy modes, $\hbar \omega_{\text{osc}} > k_B T$, quantum dynamics

Low energy modes, $\hbar \omega_{\text{osc}} < k_B T$, classical dynamics

Combining all modes

\[
t < \frac{\hbar}{k_B T}
\]
Quantum dynamics

\[
t > \frac{\hbar}{k_B T}
\]
Classical dynamics

For studying dynamics it is important to know the initial width of the phase
Relative phase dynamics

Naive estimate

\[ \delta N \sim \sqrt{N} \]

\[ \frac{N}{2} \pm \delta N \quad \frac{N}{2} \mp \delta N \]

Long Phase Coherence Time and Number Squeezing of two Bose-Einstein Condensates on an Atom Chip

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(Dated: August 27, 2006)
Relative phase dynamics

Separating condensates at finite rate

Instantaneous Josephson frequency

\[ \omega_J = \sqrt{U J} \]

Adiabatic regime \( \dot{\omega}_J < \omega_J^2 \)

Instantaneous separation regime \( \dot{\omega}_J > \omega_J^2 \)

Adiabaticity breaks down when \( \omega_J \sim 1/\tau_s \)

Charge uncertainty at this moment

\[ U \delta N^2 \sim \omega_J \sim 1/\tau_s \]

Squeezing factor

\[ \frac{\delta N}{\sqrt{N}} \sim \sqrt{\frac{1}{NU \tau_s}} \sim \sqrt{\frac{1}{\mu \tau_s}} \]
Relative phase dynamics

Quantum regime

\[
\frac{\hbar}{\mu} < t < \frac{\hbar}{k_B T}
\]

1D systems

\[
\langle e^{i\phi(t)} \rangle \sim e^{-\frac{\mu t^2}{2N\tau_s}} e^{-t/2\pi K\tau_s}
\]

2D systems

\[
\langle e^{i\phi(t)} \rangle \sim e^{-\frac{\mu t^2}{2N\tau_s}} \left( \frac{t_0}{t} \right)^{1/16TKT\tau_s}
\]

Classical regime

\[
t > \frac{\hbar}{k_B T}
\]

1D systems

\[
\langle e^{i\phi(t)} \rangle \sim e^{-\left(\frac{t}{t_T}\right)^{2/3}}
\]

\[
t_T \sim \frac{\mu K}{T^2}
\]

2D systems

\[
\langle e^{i\phi(t)} \rangle \sim \left( \frac{t_0}{t} \right)\frac{T}{8TKT}
\]
Probing spin systems using distribution function of magnetization
Probing spin systems using distribution function of magnetization

R. Cherng, E. Demler, cond-mat/0609748

Magnetization in a finite system

Average magnetization

Higher moments of $M_{tot}^z$ contain information about higher order correlation functions

$$\langle \left( M_{tot}^z - \langle M_{tot}^z \rangle \right)^2 \rangle = L \sum_{ij} \langle (M_z(i) - \langle M^z \rangle)(M_z(j) - \langle M^z \rangle) \rangle$$
Probing spin systems using distribution function of magnetization

$$\mathcal{H} = -J \sum_i [2S^x(i)S^x(i + 1) + gS^z(i)]$$

**x-Ferromagnet or polarized**

$$P(m_x)$$

$$m_x$$

$$P(m_z)$$

$$m_z$$
Distribution Functions

x-Ferromagnet

$P(m_x)$

$m_x - \langle m_x \rangle$

$n=10$
$n=50$
$n=250$

$P(m_z)$

$m_z - \langle m_z \rangle$

$n=10$
$n=50$
$n=250$
Using noise to detect spin liquids

Spin liquids have no broken symmetries
No sharp Bragg peaks

Algebraic spin liquids have long range spin correlations

\[ \langle S_i S_j \rangle = \frac{e^{iQ r_{ij}}}{|r_i - r_j|^{1+\eta}} \]

No static magnetization \[ \langle S_A \rangle = 0 \]

Noise in magnetization exceeds shot noise

\[ \langle S_A^2 \rangle = \sum_{ij} \langle S_i S_j \rangle \sim A \int_A \frac{r dr}{|r|^{1+\eta}} \sim A^{1+\frac{1-\eta}{2}} \]
Conclusions

Interference of extended condensates can be used to probe equilibrium correlation functions in one and two dimensional systems.

Analysis of time dependent decoherence of a pair of condensates can be used to probe low temperature dissipation.

Measurements of the distribution function of magnetization provide a new way of analyzing correlation functions in spin systems realized with cold atoms.