Universal Rephasing Dynamics after a Quantum Quench via Sudden Coupling of Two Initially Independent Condensates

Emanuele G. Dalla Torre, 1 Eugene Demler, 1 and Anatoli Polkovnikov 2

1Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
2Department of Physics, Boston University, Boston, Massachusetts 02215, USA

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Equilibrium systems near second-order phase transitions are the best-known examples of universal behavior in many-body physics. For these systems, universality implies that all macroscopic quantities are related to the distance from the transition point through universal critical exponents. In recent years, much effort has been dedicated to the extension of these concepts to quantum systems out of thermal equilibrium (see for example Refs. [1–3] for open systems and Ref. [4] for a review on closed systems). In particular, some authors considered closed systems that are slowly driven through a second-order phase transition and, applying the Kibble-Zurek argument, found universal scaling laws of static and dynamic quantities [5–8]. Others considered sudden quenches and predicted a universal scaling of physical observables at long times after the quench [9–12]. Here, we extend these arguments by demonstrating a universal behavior in the transient dynamics following a sudden quench, at both short and long times.

In this Letter, we consider systems that are initially prepared in the ground state of a gapless Hamiltonian and that are quenched by the sudden opening of an excitation gap $\Delta$ [see Fig. 1(a)]. In the renormalization group (RG) language, this corresponds to the sudden switching on of a relevant perturbation. The resulting dynamics is described by a strongly interacting theory and is intrinsically nonperturbative [13]. Experimentally, this can be realized, for example, by preparing two independent Bose-Einstein condensates in their respective ground states and then suddenly switching on a finite tunneling coupling $j_L$ [see Fig. 1(b)]. This opens an excitation gap for the relative-phase modes, given at a mean field level by the plasma frequency $\Delta = 2\sqrt{\mu j_L}$, where $\mu$ is the chemical potential. Beyond-mean-field effects will be extensively discussed below.

In the limit of $j_L \ll \mu$, the dynamics following the quench is dominated by the low-frequency modes and is thus expected to be universal. For a generic physical observable $C(t)$, we predict the universal contributions to have the form

$$C(t) = \left( \frac{\Delta}{\mu} \right)^{\eta} R(\Delta t).$$

(1)

Here, $\eta$ is the scaling dimension corresponding to the specific observable and $R$ is a scaling function, fixing the universal behavior of the transient many-body dynamics. As we will demonstrate, the scaling ansatz (1) is valid at times both shorter and longer than the inverse gap $1/\Delta$. This allows us to extract information about the nonperturbative long-time dynamics from the short-time dynamics, where time-dependent perturbation theory applies. A similar relation is known to hold for classical systems, with important theoretical and computational applications [14–17].

Two and three dimensions.—To substantiate our predictions, we first apply a two-mode model, valid in two and three dimensions [18]. Further conditions for the validity of this model are discussed towards the end of the Letter. By considering only the macroscopically occupied state of each condensate, we describe the sudden quench by

![FIG. 1 (color online).](image-url)

(a) Dispersion relation before (dashed line) and after (solid line) a quantum quench opening a gap $\Delta$ and involving only the low-frequency modes. (b) Physical realization: sudden coupling of two independent condensates via a uniform tunneling $j_L$. (c) Lattice model used for the numerical calculations.
\[ H(t) = \frac{\mu}{N} (\delta n_1^2 + \delta n_2^2) + \Theta(t) j_\perp (\psi_1^+, \psi_2 + \text{H.c.}). \]  

(2)

Here, \( \delta n_\alpha = \psi_\alpha^+ \psi_\alpha - N_\alpha \), with \( N_\alpha = N/2 \); \( \psi_\alpha^+ \) with \( \alpha = 1, 2 \), is a canonical bosonic operator, and \( \Theta(t) \) is the Heaviside step function. The two terms in Eq. (2) describe respectively the interactions among the atoms in the same condensate and the sudden switching on of a tunneling coupling. Results from the exact diagonalization of Eq. (2) for different values of \( j_\perp \) are shown in Fig. 2(a). Figure 2(b) shows the data collapse obtained by rescaling the time axis, and demonstrating the validity of our scaling ansatz (1), with \( \eta = 0 \).

To understand the observed scaling behavior, it is useful to introduce the phase fields \( \phi_\alpha \), canonically conjugate to \( \delta n_\alpha \), through the approximate relation \( \psi_\alpha = \sqrt{N_\alpha} e^{i\phi_\alpha} \). Moving to the relative coordinates \( \phi = (\phi_1 - \phi_2)/\sqrt{2} \) and \( n = (\delta n_1 - \delta n_2)/\sqrt{2} \), we obtain the Josephson junction Hamiltonian

\[ H(t) = \frac{2\mu}{N} n^2 - 2\Theta(t) j_\perp N \cos(\sqrt{2}\phi) \].  

(3)

The ground state and first-few excited states of the final Hamiltonian \( H(t) \) (0) are well described by the harmonic approximation \( \cos(\sqrt{2}\phi) \approx 1 - \phi^2 \). When applied to the present quench, this approximation predicts undamped oscillations with frequency \( \Delta = 2\sqrt{\mu j_\perp} \). Instead, as shown in Fig. 2, the universal dynamics displays strongly damped oscillations. The harmonic approximation breaks down because the initial state \( |n = 0 \rangle \) has a large overlap with a macroscopic number of eigenstates of the final Hamiltonian, \( N_{\text{excited}} = N\sqrt{j_\perp/\mu} \gg 1 \). For these states all higher-order Taylor components of the cosine need to be considered, highlighting the strongly interacting nature of the predicted dynamics.

In the limit of large \( N \), an alternative, nonperturbative description is available: the semiclassical approach, or truncated Wigner approximation [19–21]. Following this method (Supplemental Material [22]), we map the quantum Hamiltonian (3) to the classical equations of motion of a simple pendulum \( d^2\phi/dt^2 = -\Delta^2 \sin(\sqrt{2}\phi)/\sqrt{2} \), whose analytical solution is known. The quantum nature of the problem enters through the initial conditions, for which \( \phi \) is uniformly distributed between zero and \( \sqrt{2}\pi \). The resulting dynamics corresponds to damped oscillations with a single frequency scale \( \Delta \) and correctly reproduces the numerical solution of the full quantum model, as shown in Fig. 2(b).

One dimension.—In one dimension, the initial state is not described by two macroscopically occupied modes but rather by two “quasicondensates” with power-law correlations. Using the standard bosonization technique [23], we describe the system in terms of two phase fields \( \phi_1(x, t) \) and \( \phi_2(x, t) \) and the time-dependent Hamiltonian

\[
H = \sum_{a=1,2} H_{LL}[\phi_a] - 2 j_\perp(t) \rho_0 \int dx \cos(\phi_1 - \phi_2).
\]

Here, the first term describes the low-frequency modes of the two independent condensates as “Tomonaga-Luttinger liquids” [24] with sound velocity \( v \) and Luttinger parameter \( K \). The latter measures the ratio between the longitudinal kinetic energy and the interaction energy: \( K \rightarrow \infty \) corresponds to free bosons and \( K = 1 \) to the Tomsk-Girardeau limit. The second term of Eq. (4) describes the tunneling between the two condensates, according to the “bosonization dictionary” \( \psi_\alpha \rightarrow \sqrt{\rho_0} e^{i\phi_\alpha} \).

Introducing the relative coordinate \( \phi = (\phi_1 - \phi_2)/\sqrt{2} \), we obtain the well-known sine-Gordon model with a nonlinear term proportional to the cosine of the relative phase, \( \cos(\phi_1 - \phi_2) = \cos(\sqrt{2}\phi) \). At zero temperature, this model is characterized by a quantum phase transition between the gapless Tomonaga-Luttinger liquid \( K < 1/4 \) and a gapped phase \( K > 1/4 \). The effects of the quench in the two phases are very different. For \( K < 1/4 \), the cosine is irrelevant in an RG sense and the low-frequency dynamics is governed by the quadratic Luttinger liquid theory [i.e., the first term of the Hamiltonian (4)]. At long times, this theory predicts the appearance of universal power laws [12,25]. In contrast, for \( K > 1/4 \), the cosine is relevant and the resulting theory is strongly interacting [26]. As we will now show, in this case the dynamics is universal at both short and long times.

To study the dynamics of the problem, we first perform a series expansion in the tunneling coupling \( j_\perp \). Moving to a path-integral formulation of the problem (for an introduction see for example Refs. [28,29]), we represent the Hamiltonian (4) in terms of the real-time action,

\[
S = \int_{\gamma_K} dt \int dx \frac{K}{\pi} \left[ (\partial_x \phi^2) - (\partial_t \phi^2) \right] + 2\Theta(i) j_\perp \rho_0 \cos(\sqrt{2}\phi).
\]

(5)

Here, \( \gamma_K \) is the Keldysh contour and we switched to units where \( v = 1 \). Splitting the Keldysh contour in its forward and backward branches (\( \phi_+ \) and \( \phi_- \), respectively) and moving to their symmetric and antisymmetric
combinations (defined by \( \phi_{\pm} = \phi \pm \dot{\phi} \) and often termed “classical” and “quantum” components), we have

\[
2 \int_{\gamma_2} dt \cos(\sqrt{2} \phi) = 2 \int dt \cos(\sqrt{2} \phi_+) - \cos(\sqrt{2} \phi_-) = 4 \int dt \sin(\sqrt{2} \phi) \sin(\sqrt{2} \phi) - \sum_{\epsilon = \pm} \int dt e^{\sqrt{2} i \epsilon \phi + \sqrt{2} i \dot{\phi}}. \tag{6}
\]

Expanding \( e^{iS} \) in a Taylor series around \( j_\perp = 0 \), we obtain

\[
C_{12}(t) = \langle \cos(\sqrt{2} \phi(t)) \rangle = \text{Re} \left( e^{\sqrt{2} i \int_0^t dt \sum_{N} \frac{(ij_\perp \rho_0)^N}{N!} \times \left( \sum_{\epsilon \in \epsilon} \int_0^\infty dx e^{\sqrt{2} i (\epsilon \phi + \dot{\epsilon} \phi)} \right)^N \right). \tag{7}
\]

Note that the \( N \)-th term of Eq. \( 7 \) contains \( 2N \) continuous integrals and \( 2N \) discrete sums, whose dummy indices we will denote as \( (x_i, t_i, \epsilon_i, \dot{\epsilon}_i) \) with \( i = 1, \ldots, N \). In analogy to the equilibrium case \([30,31]\), we can interpret the first three elements as the space-time coordinates and charges of a Coulomb gas with \( N \) particles (dual to the atoms of the original problem). In contrast, the \( \dot{\epsilon}_i \) indices do not have an equilibrium analogue and rather impose a light-cone constraint (Supplemental Material \[22\]), limiting all contributing “charges” to the past-light-cone of \((x, t)\) or \( |x - x_j| < t - t_j \). This light-cone effect is generic to global quantum quenches (see for example Refs. \[25,32\]) and prevents the exact mapping to an equilibrium model.

The expectation value appearing in Eq. \( 7 \) refers to the initial state, corresponding to the ground state of two independent Luttinger liquids, and can be computed using the quadratic part of Eq. \( 5 \). After rescaling the time and space variables of integration (Supplemental Material \[22\]), we arrive to

\[
C_{12}(t) = \sum_{N} c_N \left( \frac{j_\perp}{\mu} \right)^N (\mu t)^{2N - (N + 1)/(2K)} \tag{8}
\]

\[
= \left( \frac{j_\perp}{\mu} \right)^{1/(4K - 1)} \sum_{N} c_N (\frac{2K}{j_\perp})^{(4K - 1)/4} t^{2N - (N + 1)/(2K)} \tag{9}
\]

The coefficients \( c_N \) are universal: they depend neither on the cutoff nor on \( j_\perp \).

To achieve a universal scaling of the form of Eq. \( 1 \) it is necessary to rescale the time in units of the inverse gap. For the sine-Gordon model, the excitation gap can be exactly computed using the form-factor approach \([33]\) and, in our notations, it is proportional to \( \Delta = \mu (j_\perp / \mu)^{2K/(4K - 1)} \), leading to

\[
C_{12}(t) = \left( \frac{\Delta}{\mu} \right)^{\eta} \sum_{N} c_N (\Delta t)^{2N - (N + 1)/(2K)} = \left( \frac{\Delta}{\mu} \right)^{\eta} R(\Delta t), \tag{10}
\]

with \( \eta = 1/(2K) \). Remarkably, we obtain the same “anomalous” scaling dimension \( \eta \) as in the ground state of the model \([34]\). This result corroborates with a scaling analysis \([35]\) showing that the average amount of energy per mode introduced by the quench scales to zero much faster than the gap does.

In the above derivation, we used the asymptotic values of correlation and response functions, related to the low-frequency modes, and we neglected the contributions from the high-frequency modes. These nonuniversal contributions are analytic functions of the bare parameters. As such, they can at most contribute to \( C_{12} \) a term that is linearly proportional to \( j_\perp \). In contrast, the universal contributions scale at long times as \( j_\perp^{1/(4K - 1)} \) and are therefore dominant for any \( K > 1/2 \). Remarkably, the intermediate case of \( K = 1/2 \) can be exactly solved by mapping the problem to noninteracting fermions \([27]\). Indeed, at this point, the universal and nonuniversal contributions scale in the same way and the latter are no more negligible.

At present, we were not able to compute the universal function \( R \) by the analytic resummation of the series \( 9 \). Instead, we numerically estimate its first few terms by averaging over random samplings of \((\epsilon_i, x_i, t_i)\). The resulting universal curves are shown in Fig. 3 for two different values of \( K \). Our calculations are presently limited to time scales of order \( 1/\Delta \) due to a dynamical sign problem, related to the alternating signs of the coefficients \( c_N \). To predict the universal behavior at longer times, we will now present two alternative numerical methods.

The first method extends the semiclassical approach presented above for the two-mode model. The corresponding classical equation of motion is \([13]\)

\[
\partial_t^2 \phi = \partial_x^2 \phi - \frac{2\pi}{K} j_\perp \rho_0 \sin(\sqrt{2} \phi). \tag{11}
\]

The numerical solution of this equation is in good agreement with the series expansion and extends to longer times, as shown in Fig. 4 for \( K = 5 \). Note however that Eq. \( 11 \) is characterized by a single energy scale \( \sqrt{2} \pi j_\perp \rho_0 / K \sim j_\perp^{1/2} \), matching the excitation gap of the original quantum model only for \( K \gg 1 \). As expected, the semiclassical approach breaks down for \( K \sim 1 \), where the correspondent equilibrium system approaches a quantum phase transition.

![FIG. 3 (color online). Series expansion (9). Numerical results obtained by keeping the first \( N_{\text{max}} \) terms of the series (solid curve and error bars) and analytic results (Supplemental Material \[22\]) for the first-order contribution \( N = 1 \) (dashed curve). (a) \( K = 1.2, N_{\text{max}} = 11 \); (b) \( K = 5, N_{\text{max}} = 23 \). In both cases, we used \( 10^6 \) random evaluation points of \((x_i, t_i)\).](090404-3)
Here, the energy scale is solvable, allowing us to exactly determine the Luttinger (see the caption for details).

This approach is well suited to situations, as the one under present consideration, in which the time evolution is performed with a gapped Hamiltonian. To minimize the memory requirements, we consider hard-core bosons on a ladder [see Fig. 1(c)] represented by the Hamiltonian

$$H(t) = \sum_{\alpha=1,2} \sum_{\mu} j_{\mu} (b_{\alpha,\mu}^\dagger b_{\alpha,\mu+1} + \text{H.c.}) + V n_{\alpha,\mu} n_{\alpha,\mu+1} + \Theta(t) j_{\perp} (b_{1,\perp}^\dagger b_{2,\perp} + \text{H.c.}).$$

(12)

Here, the energy scale $j_{\parallel}$ sets the chemical potential and $V < 0$ is a nearest-neighbor attractive interaction, meant to partially counterbalance the hard-core constraint [38]. For $j_{\perp} = 0$, the two independent chains are Bethe-ansatz [39] solvable, allowing us to exactly determine the Luttinger parameter [24,40]. The numerical simulation of the model (12) is presented in Fig. 5 and confirms our scaling ansatz (see the caption for details).

Discussion and conclusion.—In contrast to the above-mentioned models, actual experiments necessarily involve a finite initial temperature $T_0$ and a finite quench time $\tau$. Intuitively, one may expect the idealized model of a sudden quench to hold only as long as $\tau$ is smaller than the inverse of the energy of the highest excited state. For the Hamiltonian (3), this energy scale is $N_b s \Delta \sim N \langle j_\perp \rangle$, leading to the requirement $\tau \ll 1/(N j_\perp \Delta)$. However, as discussed in Ref. [41], the most probable excitation path does not directly connect the ground state to the highest excited state but is rather given by subsequent steps connecting neighboring levels, each with energy splitting $\Delta$. This observation indicates that the more lenient condition $\tau \ll 1/\Delta$ is sufficient to observe a universal behavior. A similar scaling argument holds for the initial temperature $T_0$, which needs to be smaller than $\Delta$. These two constraints, $\tau \ll 1/\Delta$ and $T \ll \Delta$, should be combined with the weak coupling condition $j_{\perp} / \mu = \Delta / \mu \ll 1$, requiring temperatures (ramp times) smaller than the (inverse) chemical potential by 1 order of magnitude. These conditions are in the reach of present experiments with ultracold neutral atoms (also see the Supplemental Material [22] for details of a possible experimental realization with wave interferometers of atoms on a chip [42,43]).

A final and important question regards the coupling to the modes that have been neglected in the model Hamiltonians (2) and (4). In particular, the former does not include Bogoliubov excitations and the latter neglects the coupling to the symmetric mode $\hat{\phi} = (\phi_1 + \phi_2)/\sqrt{2}$. These low-frequency modes may act as a dissipative bath for the relative-phase $\phi$ and lead to a further “rephasing.” A complete analysis of their effect is beyond the scope of the present Letter. At this stage we can only comment that, based on the analogy with the equilibrium case, one should expect the couplings to these additional modes to be irrelevant in an RG sense and thus to affect the dynamics only at time scales much larger than $\Delta$. Accordingly, the lifetime of the predicted “prethermalized” state should tend to infinity as one approaches the weak-coupling limit $j_{\perp} / \mu \to 0$.

In summary, we studied the many-body quantum dynamics of two initially independent condensates, suddenly coupled via uniform tunneling (“global quench”). In two and three dimensions, we applied a two-mode model, whereas in one dimension we mapped the problem to a gapped sine-Gordon model. Both theories are strongly interacting and lead to nonperturbative dynamics. We determined the scaling laws for expectation values of physical observables and predicted a universal time dependence, connecting short and long times. Our findings may have important implications for both numerical calculations and experiments, in which the long-time evolution is hard to achieve. They may also help to perform finite-size scaling and renormalization-group analysis in the time domain [44–47] in analogy to the established equilibrium techniques.

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FIG. 4 (color online). Truncated Wigner approach (11).
(a) Numerical results for $j_{\perp} / j_{\parallel} = 0.27$ (red, highest), 0.16 (green), 0.10 (blue), 0.060 (red), 0.036 (magenta), lowest with $K = 5$, system size $L = 400$, number of random paths $M = 400$.
(b) Data collapse according to Eq. (10), superimposed on the result from the series expansion [black curve, identical to Fig. 3(a) after the appropriate rescaling].

FIG. 5 (color online). Hard-core bosons model (12).
(a) Numerical results for $j_{\perp} / j_{\parallel} = 0.27$ (red, highest), 0.16 (green), 0.10 (blue), 0.060 (red), 0.036 (cyan), 0.022 (magenta, lowest), with $V / j_{\parallel} = -1/2$, or $K = \pi / \sqrt{2 \pi - 2 \arcsin(V/2 j_{\parallel})} \approx 1.2$. The calculations were performed using the time-evolving block decimation algorithm of Refs. [48,49] for system size $L = 100$, number of states $\chi = 100$, and took around 3 days/cycle on a single core.
(b) Data collapse according to Eq. (10), superimposed to the result from the series expansion [black curve, identical to Fig. 3(a) after the appropriate rescaling].
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[18] A homogeneous tunneling coupling in three dimensions can be realized using two different hyperfine states and suddenly turning on a frequency-matched microwave coupling.


The dynamics in the gapped phase has been studied in Ref. [27] by a quadratic expansion of the cosine term. This approach neglects the mode-coupling effect of the cosine and does not capture the rephasing dynamics.


In actual experiments with soft-core bosons, such a term is not necessary.


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