Parametric instabilities in a 2D periodically-driven bosonic system: Beyond the weakly-interacting regime

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We experimentally investigate the effects of parametric instabilities on the short-time heating process of periodically-driven bosons in 2D optical lattices with a continuous transverse (tube) degree of freedom. We analyze three types of periodic drives: (i) linear along the x-lattice direction only, (ii) linear along the lattice diagonal, and (iii) circular in the lattice plane. In all cases, we demonstrate that the BEC decay is dominated by the emergence of unstable Bogoliubov modes, rather than scattering in higher Floquet bands, in agreement with recent theoretical predictions. The observed BEC depletion rates are much higher when shaking both along x and y directions, as opposed to only x or only y. This is understood as originating from the interaction-induced non-separability along the two lattice directions. We also report an explosion of the heating rates at large drive amplitudes, and suggest a phenomenological description beyond Bogoliubov theory in this strongly-coupled regime, circular drives heat faster than diagonal drives, which illustrates the non-trivial dependence of the heating on the choice of drive.

An area of increasing interest in ultracold atoms concerns the engineering of novel states of matter using highly-controllable optical lattices [1]. In this context, a promising approach relies on applying time-periodic modulation to the system, in view of designing an effective time-independent Hamiltonian featuring the desired properties [2,4]. This Floquet engineering has emerged as a promising and conceptually straightforward way to expand the quantum simulation toolbox, enabling appealing features such as suppressed [5,6] or laser-assisted [7] tunneling in optical lattices, enhanced magnetic correlations [8], state-dependent lattices [9], subwavelength optical lattices [10], as well as synthetic dimensions [11,12], synthetic gauge fields [13,14] and topological band structures [15].

Despite these promising applications, progress in Floquet engineering has been hindered by heating due to uncontrolled energy absorption from the periodic drive. Heating is a particularly challenging problem in interacting systems, where it is known to occur due to proliferation of resonances between many-body Floquet states, not captured by the inverse-frequency expansion [11,16]. This constrains the applicability of Floquet engineering to regimes where heating is slower than the engineered dynamics. A deeper understanding of the underlying processes is essential to determine stable regions of the (large) parameter space, where the system is amenable to Floquet engineering. Additionally, interaction-mediated heating is itself an interesting nontrivial quantum many-body process. Energy absorption and entanglement production in periodically driven systems have recently been the focus of theoretical studies [16,24] and experimental investigations [5,9,25,26]. It was predicted that, whenever the drive frequency is larger than all single-particle energy scales of the problem, heating succumbs to a stable long-lived prethermal steady state, before it can occur at exponentially-long times [20,27,31]. However, this physics appears inaccessible in current bosonic cold atom experiments, which heat up much faster in practice.

A perturbative approach to understanding drive-induced heating is to analyze the underlying two-body scattering processes using Fermi’s Golden rule (FGR) [18,25,32,34]. In the weakly-interacting limit, interactions provide a small coupling between noninteracting Floquet states. However, Floquet states cannot be treated as non-interacting when the Floquet-modified excitation spectrum is itself unstable [17,19,21,35]. These instabilities indicate that heating can occur on a shorter timescale than expected from scattering theory alone.

For Bose-Einstein condensates (BEC) in optical lattices, increased heating rates arise due to the emergence of unstable collective modes. The resulting parametric instabilities can be described using a Floquet-Bogoliubov-deGennes (FBDG) approach [19], and the short-time dynamics is dominated by an exponential growth of the unstable excited modes in the BEC.
depletion time of the condensate fraction provides an experimental window to observe this and related effects. Qualitatively different behavior is expected between scattering and parametric instability rates; most notably, different power laws as a function of the interaction strength, tunneling rate and drive amplitude.

We experimentally explore these predictions in a 2D lattice subject to 1D and 2D periodic drives, by measuring decay of the BEC condensed fraction. We provide strong experimental evidence that parametric instabilities dominate the short-time dynamics over FGR-type scattering processes, which are responsible for long-time thermalization [25]. Our experiment reveals effects beyond Floquet-Bogoliubov predictions, and points out limitations in the applicability of FBdG theory.

The experiments are performed on a BEC of $^{87}$Rb atoms loaded into a square 2D optical lattice [36, 37] with principal axes along $x$ and $y$, formed by two pairs of counter-propagating laser beams with wavelength $\lambda = 814$ nm. The total atom number is $N \approx 10^5 \pm 20\%$ systematic uncertainty). Two piezo-actuated mirrors [38] sinusoidally translate the lattice along $x$ and $y$ with arbitrary amplitude, relative phase and angular frequency: $r(t) = \{\Delta x \sin(\omega t), \Delta y \sin(\omega t + \phi)\}$. We consider the effect of three drive trajectories on the heating rate: translation along $x$ only ($\Delta y = 0$), diagonal translation along $x$ and $y$ ($\Delta y = \Delta x$ and $\phi = 0$), and circular translation ($\Delta y = \Delta x$ and $\phi = \pi/2$). Therefore, the driving is 1D ($x$ only) or 2D (diagonal or circular), in a 2D system (2D array of tubes), as shown on Fig. 1. We express the amplitude $\Delta x$ in terms of the drive-induced maximum effective energy offset between neighboring lattice sites in the co-moving frame, $K_0 = \Delta E/\hbar \omega$, where $\Delta E = m \omega^2 a \Delta x$, $a$ is the lattice spacing and $m$ is the $^{87}$Rb mass. The physical displacement is $\Delta x = h K_0/\omega m$. The lattice depth $V_0$ is held constant during shaking, and is measured in units of lattice recoil energy $E_R = h^2/2m \lambda^2$. The lattice tunneling energy $J$ and the interaction strength $g$ are controlled via $V_0$. The value of $J(V_0)$ and $g(V_0)$ are calculated from the band structure, peak atom density and scattering length [39]. This results in the following periodically-driven Bose-Hubbard Hamiltonian:

$$\hat{H}(t) = \sum_{i,j} \left\{ -J \big( \hat{a}^\dagger_{i+1,j}(z) \hat{a}_{i,j}(z) + \hat{a}^\dagger_{i,j+1}(z) \hat{a}_{i,j}(z) \big) + \text{h.c.} \right\}$$

$$- \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \hat{a}_{i,j}(z) + \frac{U}{2} \hat{a}^\dagger_{i,j}(z) \hat{a}^\dagger_{i,j}(z) \hat{a}_{i,j}(z) \hat{a}_{i,j}(z)$$

$$+ \hbar \omega K_0 \left( i \sin(\omega t) + \kappa j \sin(\omega t + \phi) \right) \frac{\partial}{\partial z} \hat{a}^\dagger_{i,j}(z) \hat{a}_{i,j}(z) \right\} (1)$$

where $\hat{a}^\dagger_{i,j}(z)$ is the annihilation (creation) operator at lattice site $(i,j)$ and transverse position $z$, and $\kappa = 0$ for $x$-only and $\kappa = 1$ for 2D drives. The interaction $U$ is defined such as $U/V \sum_{i,j} \int (\hat{a}^\dagger_{i,j}(z) \hat{a}_{i,j}(z)) = g$ with $V$ the volume of the system.

In order to avoid micro-motion effects during a drive period [10], the experiments are performed at integer multiples of the period $T = 2\pi/\omega$ [39]. The drive amplitude is ramped up smoothly [10, 40, 41] in a fixed time (minimum 2 ms) corresponding to an integer number of periods [Fig. 1]. The shaking is then held at a constant amplitude for a time $\tau$. Finally, the amplitude is ramped down to zero in a few periods. Once the lattice is at rest, we turn it off in 300 $\mu$s to determine the atomic distribution. We use absorption imaging after time of flight to measure the condensate fraction as a function of $\tau$.

For most conditions, the condensate decay agrees with an exponential decay ($N(t) = N(0) e^{-\Gamma t}$), whose rate $\Gamma$ we extract from a least-square fit [39]. We measure $\Gamma$ for the three drives at different values of $\omega$, $K_0$ and $V_0$. FBdG predicts an undamped parametric instability, characterized by exponential growth of unstable modes, i.e. accelerated condensate loss. This behavior is inconsistent with the measured exponential decay of the BEC. Hence, the undamped FBdG regime does not last long compared to the typical BEC lifetime for our parameters, and interactions between the excited unstable modes and the
BEC play a significant role in the observed heating process. Nonetheless, as we discuss below, the magnitude and scaling of $\Gamma$ are well captured by a FBdG description.

Since the Floquet-renormalized hopping is $J_{\text{eff}} = J_0(K_0)$ ($J_0(K_0)$ is the $\nu$-th order Bessel function), $J_{\text{eff}} < 0$ for $K_0 > 2.4$ and the lowest Floquet band is inverted \[^5\]: the BEC then becomes dynamically unstable at $\mathbf{q} = (0, 0)$ \[^42\], but a stable equilibrium occurs at the band edge \[^33\]: $\mathbf{q} = (\pm \pi, 0)$ for a 1D drive along $x$, and at $\mathbf{q} = (\pm \pi, \pm \pi)$ for a 2D drive along $x$ and $y$ [the components of the crystal momentum $\mathbf{q}$ are measured in units of the inverse lattice spacing $a^{-1}$]. FBdG assumes an initial macroscopic occupation of these modes \[^19\]. Unless stated otherwise, for data taken at $K_0 > 2.4$ we first accelerate the BEC to the appropriate stable point while simultaneously turning on the Floquet drive \[^39\].

**Lattice depth scans:** $\Gamma(V_0)$ – A major difference between FGR and FBdG theory is the scaling of the instability rate $\Gamma$ with the hopping $J$ and interaction strength $g$. Whereas FGR predicts $\Gamma \propto (gJ)^2$, the parametric instability rate is expected to be linear ($\Gamma \propto gJ$ \[^19\]). Figure 2 shows the decay rate as a function of $gJ/\omega$, plotted as a function of $gJ/\omega$, for the 2D-diagonal drive and 1D $x$-only drive. The solid lines show the FBdG theory, and the dashed lines are linear fits to the data. The magnitude and slope are well described by the FBdG theory and is clearly inconsistent with a quadratic dependence.

**Amplitude scans:** $\Gamma(K_0)$ – Figure 3 shows the decay rate as a function of the drive amplitude $K_0$, at lattice depth $V_0 = 11E_R$ and a drive frequency $\omega = 2\pi \times 2.5$ kHz. The same $\Gamma(K_0)$ data is shown on a full range (top) and zoomed in (bottom).

Consider first the $x$-only drive. The instability growth rate predicted from FBdG theory \[^16\] \[^39\], $\Gamma = 8J_2^2(K_0)g/\omega$, agrees with the measured decay rates [Fig. 3]. In contrast, the FGR scattering approach prediction is too low by a factor of about 30, and the predicted scaling, $\propto |J_2(K_0)|^2$ to leading order, does not describe the data as well. The agreement with the FBdG theory, despite evidence for effects beyond simple undamped parametric instability, is consistent over a range of parameter space in $K_0$, $\omega$ and $V_0$. As expected, the decay dramatically increases when $J_{\text{eff}} < 0$ ($K_0 > 2.4$) for $\mathbf{q} = (0, 0)$, while it is partially stabilized when accelerating the BEC to $\mathbf{q} = (\pi, 0)$. We note that significant heating occurs during the drive turn-on and acceleration phase for the $\mathbf{q} = (\pi, 0)$ data, resulting in partial BEC losses.

For the two 2D drives (circular and diagonal) we observe heating rates that follow roughly the same func-
tional form as the 1D drive, but about 3× larger. Additionally, the sudden increase in the decay rate $\Gamma$ for the 2D drives consistently occurs at a critical amplitude $K_0$ below 2.4. At $11E_R$, $K_0 \approx 2.15$ [Fig. 3]. Above $K_0$, both 2D rates increase dramatically beyond any prediction and the circular rate increases faster than the diagonal rate. This drive dependence and drastic rate increase for $K_0 < K_0 < 2.4$ suggest effects beyond FBdG theory, distinct from the simple parametric instability, and are discussed at the end of this Letter. For $K_0 \gtrsim 2.4$, the rates are essentially unmeasurable since $\Gamma > \omega$. As with the 1D drive, accelerating the BEC to $q = (\pi, \pi)$ partially stabilizes the decay, just enough to be measurable.

Frequency scans — $\Gamma(\omega)$: FBdG theory predicts distinct behavior at low and high drive frequencies. In the low-frequency regime, the momenta of the maximally-unstable mode $q_{\text{num}}$ evolve as one increases the drive frequency, until it saturates at the Bogoliubov band edge at $q_{\text{num}} = \{ (\pi, 0), \} \{ (\pi, 0), (0, \pi) \}$, and $\{ (\pi, \pi) \}$ for $x$-only, diagonal and circular drive, respectively. The saturation frequency $\omega_c = E_{\text{Bog}}(q_{\text{num}})$ marks the onset of the high-frequency regime [19]. Note that unlike in 1D lattices [19] the energy of the maximally unstable mode can be lower than the full effective bandwidth [39]. The rate is predicted to increase quasi-linearly for $\omega \leq \omega_c$ while $\Gamma \propto \omega^2$ for $\omega \geq \omega_c$ [19] [25], resulting in a cusp in the rate at $\omega_c$.

Figure 4 shows the experimental values for $\Gamma(\omega)$ compared with FBdG theory. Since there is no observed rate explosion for the $x$-only drive, we use the cusp in $\Gamma_x(\omega)$ to calibrate our experimental value for $g$, which agrees to within 20% with an estimate calculated from the lattice parameters [39]. Using this value of $g$, the prediction for the diagonal drive $\omega^{\text{Diag}}$ matches the experiment. While FBdG theory predicts the same $\omega_c$ for circular and $x$-only drives, the measured cusp for the circular drive lies between the cusps of the two linear drives.

The observed behavior qualitatively fits FBdG theory, with rates generally higher than predicted for $\omega \sim \omega_c$. For the $x$-only drive, the measured rates are slightly above the prediction below $2\pi \times 1.5$ kHz, and the agreement is excellent above $2\pi \times 1.5$ kHz, as observed with $\Gamma(\omega)$ at $2\pi \times 2.5$ kHz [Fig. 3]. The 2D rates show a larger discrepancy at low frequencies, and a decent quantitative agreement for $\omega \gtrsim 2\pi \times 2$ kHz. This is related to the rates explosion appearing for $K_0 > K_0$. As we discuss below, the observed value of $K_0$ increases with the frequency. This implies a similar rate explosion should happen when decreasing $\omega$ at fixed $K_0$. This is especially visible with the diagonal drive [Fig. 2]; for $\omega < 1$ kHz the data abruptly departs from the prediction. This increased rate at low frequencies for 2D drives is likely responsible for the discrepancy between experiment and theory. The presence of the cusps in the rate explosion region is still expected since, for $\omega$ low enough, some modes are energetically in-accessible, and the limit $\Gamma(\omega \to 0) \to 0$ must be fulfilled.

Rates explosion — Beyond a critical amplitude $K_0$, we observe a sudden increase of the 2D-driven heating rate [Fig. 3]. The dependence of $K_0$ on the frequency for circular driving at $V_0 = 11E_R$ is shown on Fig. 5. We observe that $K_0 \to 2.4$ as $\omega \to \infty$, suggesting that the giant instabilities arise from a finite-frequency effect. Assuming they result from an interplay of correlated physics beyond the Bogoliubov regime and the breakdown of the infinite-frequency approximation inherent to our Floquet analysis [2–4], we make the following scaling argument. Viewing the system as an effective Bose-Hubbard model, the strongly-correlated regime is reached for $g/J_{\text{eff}} \gtrsim 1$. On the other hand, we note that corrections to the infinite-frequency Floquet Hamiltonian scale as $J/\omega$. We make the phenomenological observation that the dimensionless ratio $(g/J_{\text{eff}})(J/\omega)$ should be relevant to a combination of beyond-mean-field and finite-frequency effects. When $J_{\text{eff}}$ is low in all lattice directions, $(g/J_{\text{eff}})(J/\omega)$ is large and, therefore, these effects should be large. The simple scaling relation $g/\omega J_{\text{eff}}(K_0) = 1$ gives $K_0(\omega) = J_{\text{eff}}^{-1}(g/\omega)$, and is shown as the red line in Fig. 4 along with the experimental data. The agreement is surprisingly good for such a simple argument, which supports the intuition that the rate explosion is due to the ratio $g/J_{\text{eff}}$ becoming so large that the system’s characteristic timescale is on the order of a drive period. The quantum many-body nature of the rates explosion calls for more extensive study, that promises new insights into periodically-driven strongly-correlated quantum lattice systems.

We present a detailed investigation of heating for in-

![FIG. 4. Heating rates vs drive frequency. $\Gamma(\omega)$ for the three drive trajectories at $K_0 = 1.25$ and $V_0 = 11E_R$. The Floquet-Bogoliubov-deGennes theory is shown for each trajectory as dashed lines. The theoretical cusps positions are marked as vertical black dashed lines. Rate explosion occurs at low frequencies when $K_0 > K_0(\omega)$ [cf. Fig. 3], represented as the light red zone.](image-url)
interacting bosons in a periodically driven 2D lattice. The observed heating rates are substantially larger than expected from a scattering theory based on Fermi’s golden rule, and scale as expected for interaction-driven parametric instabilities. The lack of observed exponential growth of unstable modes suggests that interactions between these excited modes and the BEC, not captured by FBdG theory, play an important role in the dynamics. Nonetheless, the linear scaling of the condensate loss rate with $gJ/\omega$, indicative of direct, interaction-induced instabilities, has been recently investigated with bosonic condensates.

We note that complementary signatures of parametric instabilities have been recently investigated with bosonic atoms in periodically-driven 1D optical lattices [43]. The observed heating rates are substantially larger than expected from current theories. Altogether, these instabilities arise from collective modes and involve coherent processes, unlike scattering in a purely FGR approach. In addition, for 2D driving, there exist regions where the heating is even larger than predicted by FBdG, which is not explained by current theories. Altogether, our observations provide important insight into the leading heating mechanism in interacting Floquet systems, a valuable knowledge for future many-body Floquet engineering schemes.

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FIG. 5. $K_0^c(\omega)$: Critical drive amplitude measured for various frequencies $\omega$ at $V_0 = 11E_R$. For $K_0 > K_0^c$ (light red region), the heating rate for both 2D drive trajectories increases dramatically, and we observe $K_0^c \rightarrow 2.4$ at large $\omega$. A simple relation captures this feature [red line].
The condensate fraction can decay either by direct Floquet-driven loss or by heating due to relaxation of energetic excitations. The latter mechanism occurs over a thermalization time scale, and can be probed by observing relaxation of out of equilibrium states in an undriven, static lattice. Special attention to the drive turn off is required to avoid unwanted excitation due to micro-motion during a drive period. Abruptly turning off the drive induces a kick large enough to create a significant out-of-equilibrium population by inter-band excitation, rough the Floquet drive alone. A special attention to the drive turn on/off is therefore required to ensure a correct interpretation of the data.

To determine the relaxation time scale and test for this additional condensate loss mechanism, we measured the evolution of the condensed fraction when holding the atomic cloud in a static lattice, immediately after an abrupt stop of the drive. While the condensed fraction is initially unchanged, upon letting the static system evolve for a time $t_{\text{hold}}$, we observe a subsequent decrease of the condensed fraction as excited atoms thermalize with the rest of the sample. Fig. 6 shows an example of thermal-
The condensed fraction at $t_{\text{end}}$ ($t_{\text{hold}} = 0$) and at $t = t_{\text{end}} + 6\text{ ms}$ ($t_{\text{hold}} = 6\text{ ms}$), as a function of the end drive phase, for a $2\pi \times 2\text{ kHz}$ diagonal (2D) drive. For an abrupt stop (blue), a large population can be transferred out of the BEC for some end phases. This results in a large thermalization event and a clear drop of the condensed fraction. The effect is minimal when stopping the drive such that $q(t)$ is a smooth function. If, however, the drive is stopped by ramping down the amplitude (green), no condensed fraction decay can be observed at any phase.

**Experimental considerations**

**Accelerating the BEC to the band edge.** – In order to accelerate our BEC from $q = (0,0)$ to a desired $q = (q_x, q_y)$, we apply a constant force $F = \dot{q}$ for a fixed time in the lattice plane. The force is generated with a constant magnetic gradient, acting on the BEC in the $|F = 1, m_F = -1\rangle$ ground state. Bias coils in the three spatial directions control the gradient direction.

The BEC becomes unstable for quasimomenta about halfway to the band edge: this is the well-known static dynamical instability [42]. On the other hand, as mentioned in the main text, ramping up the drive amplitude beyond $K_0 = 2.4$ reverses the band smoothly as $J_{eff}$ becomes negative: $q \sim (0, 0)$ becomes unstable and the band edges (or corners, depending on the drive trajectory) become stable. In order keep the BEC in a stable region (in a effectively static dynamical stability sense) at any given time, we synchronize the BEC acceleration with the ramping on of the drive, such that the BEC crosses the static instability point $q = 0.6\ \pi/a$ when $K_0 = 2.4$.

We accelerate the BEC to $q = (\pi, \pi)$ for the two 2D drives (diagonal and circle) and to $q = (\pi, 0)$ for the $x$-only 1D drive, since these become stable whenever $K_0 > 2.4$.

**Calibration of tight-binding parameters.** – The lattice depth is calibrated via Raman-Nath diffraction. In the tight-binding limit, the tunneling rate is derived through the modeled 1D dispersion as

$$J \equiv \frac{E(q = \pi/a) - E(q = 0)}{4}$$

(2)

The on-site interaction $g$ is calibrated from the $x$-only drive cusp (see Fig. 4). The point at which the rates go from increasing to decreasing, $\omega_c^{\text{on}}$, is given by

$$\omega_c^{\text{on}} = \sqrt{4J_{\text{eff}}(4J_{\text{eff}} + 2g)}.$$ 

(3)

Knowing $J$, the experimental value of $\omega_c^{\text{on}}$ offers a calibration for $g$. For $V_0 = 11E_R$, $J = 50\text{ Hz}$ and the measured value of $\omega_c^{\text{on}} = 444\text{Hz}$ gives $g = 700\text{ Hz}$. As an additional check, this value of $g$ is then used to predict the diagonal
drive cusp, \( \omega_{c, \text{Diag}} = \sqrt{8J_{\text{eff}}(8J_{\text{eff}} + 2g)} = 655 \) Hz. The observed value of \( \sim 650 \) Hz is in good agreement with this prediction.

The interaction strength \( g \) depends upon the atom number, the dipole trap and the lattice parameters. To confirm that the experimental calibration matches these known experimental parameters, we also estimate \( g \) through tight binding and Thomas-Fermi assumptions: in the lattice plane, the wavefuntion \( \psi(x) \) is taken to be well approximated by a Mathieu function, while we use a Thomas-Fermi profile in the tube direction. The interaction energy \( g \propto \int \int \frac{a}{4\pi} |\psi(r)|^4 d^2 r \) is then calculated from the known experimental parameters, including the density profile due to the dipole trap (frequencies \( \{ \omega_x, \omega_y, \omega_z \} = \{ 11, 45, 120 \} \) Hz). For \( V_0 = 11E_R \) we find \( g = 850 \) Hz, similar to the calibrated value of 700 Hz. Note that the systematic 20% uncertainty in the atom number can easily explain the small offset between estimation and calibration.

**Bandwidths** – It is important to note that \( \omega_c \) is in general different from the effective bandwidth \( B \). For a 2D (diagonal or circular) drive in our 2D lattice,

\[
B_{2D}^{2D} = \sqrt{8J|J_0(K_0)|} \times (8J|J_0(K_0)| + 2g) \tag{4}
\]

and, with a 1D drive in the 2D lattice,

\[
B_{1D}^{2D} = \sqrt{4J(|J_0(K_0)| + 1)} \times (4J(|J_0(K_0)| + 1) + 2g), \tag{5}
\]

which is in general different from \( \omega_c \), as observed in the main text: only in the case of the diagonal drive do we find that the maximally unstable mode had the maximum ground band energy, and therefore \( \omega_c = B \).

**Background rates.** – All theoretical plots take into account the background decay rate, predominantly due to lattice photon scattering. We experimentally determine this by setting \( K_0 = 0 \) and measuring the resulting rate with the same procedure as in the main text. This constant rate \( y_0 \sim 1 \text{s}^{-1} \) for \( V_0 = 11E_R \) is then added to the FBdG formula for comparison with the experimental data.

**Extracting the Instability Rates**

**Rates extraction** – Our data consists of series of measured condensed fractions after various driving times. A time series typically presents an exponential-looking decay. An example for such decay is given in figure 8. Since we focus on early the time decay rate, greater weight is given to earlier data points. We fit to an exponential with no offset (2 fit parameters): \( f(t) = Ae^{-\Gamma t} \), where \( A \) is the \( t=0 \) condensed fraction (typically \( A \geq 0.5 \)). The rates presented in this manuscript are the extracted fit parameters \( \Gamma \), and the uncertainties correspond to \( \pm 1 \) standard deviation.

For these rates to be directly compared to the BdG prediction, additional considerations must be taken. First, let us consider a maximally unstable Bogoliubov mode with an amplitude predicted to grow as \( e^{\Gamma t} \). The experiment will actually detect a rate \( 2\Gamma \), as it measures an amplitude squared (typically, the number of atoms in the unstable mode). Second, since the experiment measures how many atoms leave the BEC (to populate the modes) per unit of time, it is sensitive to the number of simultaneous maximally unstable modes, as each is a decay channel. If two modes are equally and maximally unstable, as is possible in 2D, then an additional factor of 2 is needed in the theory to compare to the experiment. This multiple-modes factor is 1 for the \( x \)-only drive, and 2 for the circle and diagonal drives.

All these extra factors were added to the theory plots throughout this paper: in total, the \( x \)-only drive theory is \( 2\times \) and the circle and diagonal drives are \( 4\times \) larger than the bare BdG rates predicted in [19].

**Difficulties associated with dynamical rates** – In the main text, we compared the instability rates measured in the experiment to those predicted by FBdG theory. Here, we elaborate on some intrinsic difficulties in the procedure which may affect the extracted values. As explained in Ref. [19], for drive frequencies below the effective drive-renormalized Floquet-Bogoliubov...
bandwidth, there exists an entire manifold of resonant modes. While all of them contribute to expectation values of observables at very short times, only the maximally-unstable mode \( q_{\text{num}} \) dominates the long-time BdG dynamics, and the rate associated with \( q_{\text{num}} \) sets the parametric instability rate. Thus, at any finite time, the FBdG dynamics is in a crossover between these two regimes, which shrinks exponentially with time. Yet the time-width of this crossover also depends on the drive frequency: the higher the frequency, the smaller the instability rate and the longer it takes for the exponential behavior to become visible.

When extracting the rates from data, effects due to this crossover become relevant. To test this, we performed exact numerical simulations of the BdG equations of motion, and computed the dynamics of the excited fraction of atoms \( n_q(t) \) over a finite number of driving cycles, which increases suitably with the drive frequency. We then extracted the instability rates using least-square fitting as the slope of \( \log n_q(t) \) over the last eight driving cycles, to maximally eliminate transient effects. A comparison between the numerically-extracted rates and the analytic theory prediction is shown in Fig. 9 for the three types of drives. Note that the agreement becomes worse at larger \( \omega \), since this decreases the rate and pushes the exponential regime to later times. This is a source of error, which is certainly relevant for the experimental determination of the rates.

Additionally, in the experiment there are strong beyond Bogoliubov effects, not captured by FBdG theory. Due to the nonlinearity of the Gross-Pitaevskii equation which leads to saturation of the condensate depletion, the exponential BdG regime mentioned above crosses over into a third, scattering-dominated regime. In this regime, the population transferred coherently to the maximally-unstable modes by the parametric resonance starts decaying into the surrounding finite-momentum modes at high energy, leading to uncontrolled irreversible heating. This suggests that the instability rates can change in time. This additionally obfuscates the comparison of the experiment with the FBdG theory.

**HEATING DYNAMICS OF THE TRUNCATED WIGNER APPROXIMATION**

While FBdG theory is valid in the short-time regime of the dynamics, it has some serious deficiencies. Perhaps the most notable of these, when it comes to out-of-equilibrium dynamics, is the lack of particle-number conservation: the condensate is assumed to be an infinite reservoir which supplies particles to indefinitely increase the occupation of pairs of modes with finite and opposite momenta. In equilibrium, this description works well and captures the physics in the superfluid phase. Away from equilibrium, however, condensate depletion processes such as the parametric instabilities studied in this work lead to significant depletion of the BEC and the mean-field Bogoliubov description ultimately breaks down under typical observation times.

Particle conservation is obeyed in the Truncated Wigner Approximation (TWA), which also includes nonlinear interactions modeling collisions between Bogoliubov quasiparticles, and is capable of describing thermalization at later stages, due to the continuous pumping of energy into the system.

The starting point for the TWA is the Gross-Pitaevskii equation which, in the co-moving real-space frame, reads \( (\hbar = 1) \).
where $a_r(t)$ models the bosonic system at time $t$ and position $\mathbf{r} = (x, y, z)$. The kinetic energy reflects the lattice degrees of freedom in the $(x, y)$-plane, and the continuous transverse mode along the $z$-axis. The periodic drive is in the $(x, y)$-plane with frequency $\omega$ and amplitude $K_\omega$. $\kappa_d = 0$ for the $x$-only drive and $\kappa_d = 1$ for both 2D drives. $\phi$ is the relative drive phase between $x$ and $y$ ($\phi = 0$ for diagonal drive and $\phi = -\frac{\pi}{2}$ for circular drive). Finally, the on-site interaction strength is denoted by $U$ (such that $U/V \sum_{i,j} \int d\mathbf{z} |a_r|^2 = g$).

Since at time $t = 0$ the system forms a BEC of $N_0$ atoms in a volume $V$, assuming a macroscopic occupation $n_0 = \sqrt{N_0/V}$ in the uniform ($q = 0$) condensate, the field $a_r$ can be decomposed as

$$a_r = n_0 + \frac{1}{\sqrt{V}} \sum_{q \neq 0} u_q \gamma_q e^{-iq \cdot r} + v_q^* \gamma_q^* e^{iq \cdot r},$$

where $u_q$ and $v_q$ are the Bogoliubov modes which solve the time-independent BdG equations at $t = 0$ [19]. Here, $\gamma_q$ is a complex-valued Gaussian random variable [associated with the quantum annihilator $\gamma_q$ of Bogoliubov modes] with mean and variance set by the corresponding quantum expectation values in the Bogoliubov ground state [46].

Hence, in the TWA one draws multiple random realizations of $\gamma_q$, each of which corresponds to a different initial state. One then evolves every member of this ensemble according to Eq. (6), computes the observable of interest, and takes the ensemble average (7) in the end. For instance one can compute the total number of excited atoms as

$$n_{ex}(t) = \frac{1}{V} \sum_{q \neq 0} |a_q(t)|^2,$$

which, due to particle number conservation also reflects the dynamics of the condensate depletion.

In general, we expect that the condensate depletion curve shows two types of behavior: at short times FBdG theory applies and $n_{ex}(t) \sim \exp(2FT)$ grows exponentially in time. Hence, the condensate depletion curve $|a_q=0(t)|^2 = V(n_0 - n_{ex}(t))$ is concave. At long times, nonlinear interaction effects in the GPE become important, leading to saturation, and the curvature of condensate depletion changes sign. Therefore, in the long-time regime the curve is concave. The opposite behavior is true for the evolution of the excitations $n_{ex}(t)$. The curvature of the experimental data, cf. Fig. 8 suggests that the system enters well into the long-time regime. Yet, the measured decay rates appear consistent with the short-time Bogoliubov theory [main text].

To shed light on this intriguing observation, we perform TWA simulations on a periodically-driven homogeneous system in $(2+1)$-dimensions, and extract the short-time and long-time rates from the numerical data. We use a comparison with the BdG simulations, to separate the short-time regime (where agreement between BdG equations of motion and TWA is expected) from the longer-time regime [10]. For the sake of comparison with experiments, we fit the long-time TWA growth to an exponential, even though we find that it follows a more complicated functional form.

Figure 10b shows a scan of the TWA rates over the effective interaction parameter $g$. We find that both the short-time and long-time rates are of similar strength. More importantly, they do not show a quadratic scaling in $g$, as predicted by Fermi’s Golden Rule. This behavior is consistent with the experimental observations. Note the mismatch between FBdG theory [black] and the short-time BdG simulations [blue], which arises since the most unstable mode does not yet dominate the dynamics at short times [see Fig. 9 and corresponding discussion]. Indeed, we find an excellent agreement between BdG numerics and FBdG theory if we extract the rates from the long-time regime. The short-time BdG rates agree qualitatively with the short-time TWA rates, as expected from the agreement seen in Fig. 10a. The rates are extracted from a least-square fit over the last 5 consecutive driving cycles of short-time region of agreement between BdG and TWA. Since the rates are dynamical, i.e. change depending on the time-window used to extract them, the curves in Fig. 10b are not smooth.

We also did a frequency and amplitude scans of the long-time TWA rates to look for signatures of the Bessel function $J_2(K_0)$, and the cusp at the critical frequency $\omega_c$, as expected from FBdG theory and found experimentally. Unfortunately, we do not see clear signatures of such behaviors in our TWA simulations. Thus, we cannot conclude that the TWA captures the long-time thermalization dynamics of driven bosonic cold atom systems accurately. More interestingly, the rates explosion [see main text] is also beyond the TWA dynamics, suggesting that quantum effects, such as loss of coherence, are important for describing this phenomenon. Another possible reason for disagreement is the single band approximation, as its validity for the Floquet system has not been fully understood so far.
FIG. 10. Rate comparison: TWA vs. BdG Left panel: Numerical simulation of the excitations growth using TWA (solid lines) and FBdG (dashed lines). The TWA curves change curvature beyond the regime of validity of BdG. Right panel: excitations [i.e. condensate depletion] growth rate against effective interaction strength $g$ for FBdG theory (black), BdG short-time evolution (blue), TWA short-time evolution (red) and TWA long-time evolution (green). The parameters are $K_0 = 2.1$, $\omega/J = 20.0$ and $n_0 = 50.0$. We used a system of $80 \times 80 \times 101$ momentum modes in the $(x,y,z)$-direction, respectively. The TWA data is averaged over 50 independent realizations and the error bars (shaded area) is computed using a bootstrapping approach.