

Exactly solvable case of a one-dimensional Bose–Fermi mixture

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We consider a one-dimensional interacting Bose-Fermi mixture with equal masses of bosons and fermions, and with equal and repulsive interactions between Bose-Fermi and Bose-Bose particles. Such a system can be realized in experiments with ultracold boson and fermion isotopes in optical lattices. We use the Bethe-ansatz technique to find the ground state energy at zero temperature for any value of interaction strength and density ratio between bosons and fermions. We prove that the mixture is always stable against demixing. Combining exact solution with the local density approximation, we calculate density profiles and collective oscillation modes in a harmonic trap. In the strongly interacting regime, we use exact wave functions to calculate correlation functions for bosons and fermions under periodic boundary conditions.

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Recent developments in the cooling and trapping of atomic gases open exciting opportunities for experimental studies of interacting systems under well-controlled conditions. Using Feshbach resonances [1] and/or optical lattices [2,3], it is possible to reach strongly interacting regimes, where correlations between atoms play a crucial role. The effect of interactions is most prominent for low dimensional systems, and recent experimental realization [4,5] of a strongly interacting Tonks–Girardeau (TG) gas of bosons opens new perspectives in experimental studies of strongly interacting systems in one dimension [6]. A different line of research, which has attracted considerable attention both theoretically [7] and experimentally [8,9], is the study of atomic mixtures. One-dimensional (1D) Fermi gases have been theoretically well investigated [10], since they correspond to interacting electrons in solid state quasi-1D systems. On the other hand, 1D Bose–Fermi (BF) mixtures did not attract equal attention until recently [11–14], when it became possible to realize such systems in experiments with cold atoms. Several properties of such systems have been investigated so far, including phase separation, fermion pairing and charge density wave (CDW) formation. However, most of these investigations relied on the mean-field approximations or the Luttinger liquid (LL) formalism. The mean-field approximation is known to be unreliable in 1D, and the LL approach does not allow the calculation of LL parameters in the strongly interacting regime. In this paper, we study BF mixtures in the regime, where an exact Bethe-ansatz solution is available. We use the exact solution to calculate the ground state energy and investigate phase separation and collective modes. In the strongly interacting regime, we derive an analytical determinant formula, which allows one to calculate correlation functions at all distances numerically for a polynomial time in the system size. The exactly solvable case considered in this paper is relevant to current experiments, and can be used as a benchmark to check the validity of different approaches.

A 1D interacting BF mixture is described by the Hamiltonian

$$H = \int_0^L dx \left(\frac{\hbar^2}{2m_b} \partial_x \Psi_b^\dagger \partial_x \Psi_b + \frac{\hbar^2}{2m_f} \partial_x \Psi_f^\dagger \partial_x \Psi_f \right) + \int_0^L dx \left(\frac{1}{2} g_{bb} \Psi_b^\dagger \Psi_b^\dagger \Psi_b \Psi_b + g_{bf} \Psi_b^\dagger \Psi_f^\dagger \Psi_f \Psi_b \right). \quad (1)$$

Here, Ψ_b, Ψ_f are boson and fermion operators, m_b, m_f are the masses, and g_{bb}, g_{bf} are boson-boson and boson-fermion interaction strengths. An array of such 1D tubes can be realized using a strong optical lattice in y, z directions [4,5,15]. Alternatively, one can create a single Bose Einstein Condensate (BEC) in a 1D box potential using the techniques of Ref. [16]. The model (1) is exactly solvable, when

$$m_f = m_b \equiv m, \quad g_{bb} = g_{bf} \equiv g. \quad (2)$$

The first condition is approximately satisfied for isotopes of atoms. Some of the promising candidates are $^{171}\text{Yb}+^{172}\text{Yb}$ (Ref. [18]), $^{39(41)}\text{K}-^{40}\text{K}$, (Ref. [17]), and $^{86(84)}\text{Rb}-^{87(85)}\text{Rb}$ [19]. Alternatively, if one uses an additional optical lattice along the x direction with filling factors smaller than one, then Eq. (1) is an effective Hamiltonian describing this system with the masses determined by the tunneling [5]. Thus, one can simulate Eq. (1) using already available degenerate mixtures [8]. We point out that to satisfy the second condition in Eq. (2), it is sufficient to have equal (positive) signs for the two scattering lengths, but not necessarily their magnitudes. Well away from confinement induced resonances [20], 1D interactions are given by $g_{bb}=2\hbar\omega_{b\perp}a_{bb}, g_{bf}=2\hbar\sqrt{\omega_{b\perp}\omega_{f\perp}}a_{bf}$, where $\omega_{b\perp}, \omega_{f\perp}$ are radial confinement frequencies, and a_{bb}, a_{bf} are three-dimensional 3D scattering lengths. For a fixed value of a_{bb}/a_{bf} , one can always choose the detuning of the optical lattice laser frequencies in such a way that $g_{bb}=g_{bf}$. Additionally, all interactions can be tuned using available BF Feshbach resonances [1].

Model (1) under conditions (2) has been considered in the literature before [21], but its properties have not been investigated in detail. Similar to the case of bosons [22], a dimensionless parameter that controls the strength of interactions is $\gamma=mg/(\hbar^2n)$, with n being the total density of particles. $\gamma \gg 1$ corresponds to the strongly interacting mixture. The

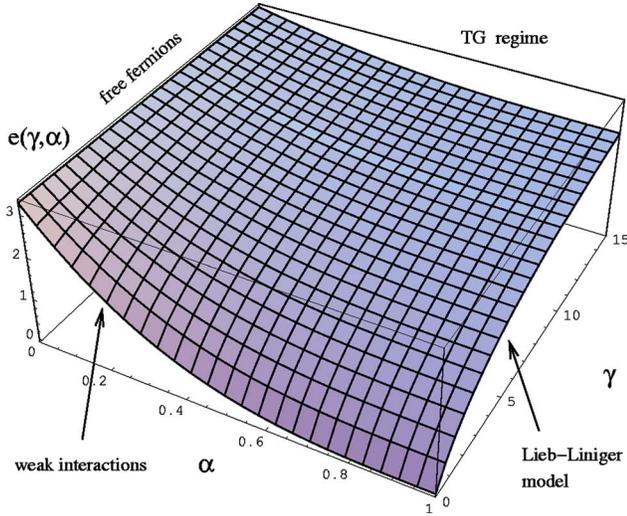


FIG. 1. (Color online) Energy of the ground state is given by $E=e(\gamma, \alpha)\hbar^2 N^3/(2mL^2)$, where $\gamma=mg/(\hbar^2 n)$, and $\alpha=N_b/N$ is the boson fraction. When $\alpha=0$, the system is purely fermionic, and the energy does not depend on interactions. When $\alpha=1$, the system is purely bosonic, and numerically obtained energy coincides with the result of [22]. If $\gamma=0$, bosons and fermions do not interact, and $e(\gamma, \alpha)=(\pi^2/3)(1-\alpha)^3$.

ground state energy of the model (1), (2) with periodic boundary conditions is related to the solution of a system of integral equations

$$2\pi\rho(k) = 1 + \int_{-B}^B \frac{4c\sigma(\Lambda)d\Lambda}{c^2 + 4(\Lambda - k)^2}, \quad (3)$$

$$2\pi\sigma(\Lambda) = \int_{-Q}^Q \frac{4c\rho(\omega)d\omega}{c^2 + 4(\Lambda - \omega)^2}. \quad (4)$$

Here $c=mg/\hbar^2$, and Q, B are related to densities via

$$N/L = \int_{-Q}^Q \rho(k)dk, \quad N_b/L = \int_{-B}^B \sigma(\Lambda)d\Lambda, \quad (5)$$

where N_b is the number of bosons, N is the total number of particles, and L is the length of the system. The ground state energy can be expressed as

$$E/L = \frac{\hbar^2}{2m} \int_{-Q}^Q k^2 \rho(k)dk. \quad (6)$$

One can solve Eqs. (3)–(5) numerically and calculate the energy as a function of γ and relative boson density $\alpha = N_b/N$. After rescaling $E=e(\gamma, \alpha)\hbar^2 N^3/(2mL^2)$, where $e(\gamma, \alpha)$ is shown in Fig. 1. In the limit $\gamma \gg 1$, one can derive the large γ expansion of the energy $e(\gamma, \alpha) = (\pi\gamma)^2/3(\gamma + 2(\alpha + \sin(\pi\alpha)/\pi))^2$. Chemical potentials for bosons and fermions are given by $\mu_b = \partial E/\partial N_b$, $\mu_f = \partial E/\partial N_f$, where $N_f = N - N_b$. Using the exact solution, one can analyze demixing instabilities [11,12] for repulsive BF mixtures. In the absence of an external potential, the BF mixture is stable if the compressibility matrix $\partial\mu_{b(f)}/\partial n_{b(f)}$ is positively defined [here, $n_{b(f)}$ is a boson (fermion) density]. In our system, we verified

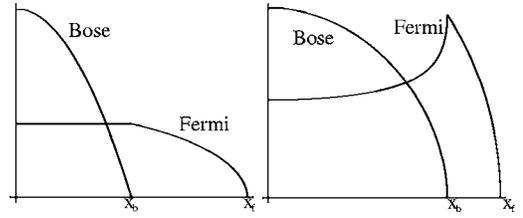


FIG. 2. Densities of Bose and Fermi gases in weakly interacting (left, $\gamma_0=0.18$) and strongly interacting ($\gamma_0 \gg 1$) regimes. The overall number of bosons equals the number of fermions.

numerically that this condition is satisfied for all α and γ . This proves that the BF mixture under conditions (2) is stable against demixing for any strength of interaction. This is in contrast to mean-field approximation which predicts demixing for strong enough interactions [11]. Although an exact solution is available only under conditions (2), small deviations from these should not dramatically change the energy $e(\gamma, \alpha)$. Therefore, we expect the 1D mixtures to remain stable to demixing in the vicinity of the integrable line (2) for any interaction strength.

Using numerically obtained $e(\gamma, \alpha)$ and local density approximation (LDA), one can investigate the behavior of the system in an external field. LDA is expected to be valid for slowly varying external potential. We will assume that the confinement frequency for bosons and fermions is the same. Within LDA, density distributions in the region of coexistence are governed by a set of equations $\mu_f(n_b^0(x), n_f^0(x)) + m\omega_0^2 x^2/2 = \mu_f(n_b^0(0), n_f^0(0))$ and $\mu_b(n_b^0(x), n_f^0(x)) + m\omega_0^2 x^2/2 = \mu_b(n_b^0(0), n_f^0(0))$. It turns out that this set of equations cannot be satisfied for the whole cloud, and consequently a phase separation occurs in the presence of such external potential. The BF mixture is present in the central part, but the outer sections consist of Fermi gas only. In the weakly interacting limit, this can be interpreted as an effect of the Fermi pressure [9]. While bosons can condense to the center of the trap, Pauli principle pushes fermions apart. As interactions get stronger, the relative distribution of bosons and fermions changes, and Fig. 2 contrasts the limits of strong and weak interactions. For strong interactions, the Fermi density shows strong nonmonotonous behavior.

Recent experiments [15] demonstrated that collective oscillations of 1D gases provide useful information about interactions in the system. Within the region of coexistence of bosons and fermions, such oscillations can be described by four hydrodynamic equations [23]

$$\frac{\partial}{\partial t} \delta n_{b(f)} + \frac{\partial}{\partial x} (n_{b(f)} v_{b(f)}) = 0, \quad (7)$$

$$m \frac{\partial}{\partial t} \delta v_{b(f)} + \frac{\partial}{\partial x} \left(\mu_{b(f)} + V_{ext, b(f)} + \frac{1}{2} m v_{b(f)}^2 \right) = 0. \quad (8)$$

In certain cases, analytical solutions of hydrodynamic equations are available [23,24] and provide the frequencies of collective modes. When an analytic solution is not available, the ‘‘sum rule’’ approach has been used [23–26]. The disadvantage of the latter approach is an ambiguity in the choice

of the multipole operator that excites a particular mode, especially for multicomponent systems [26]. Here, we develop an efficient numerical procedure for solving the hydrodynamical equations in 1D, which does not involve additional sum rule approximation. After linearization, a system of hydrodynamic equations can be written as

$$-m\omega^2 \begin{bmatrix} \delta\mu_b(x) \\ \delta\mu_f(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial\mu_b}{\partial n_b} & \frac{\partial\mu_b}{\partial n_f} \\ \frac{\partial\mu_f}{\partial n_b} & \frac{\partial\mu_f}{\partial n_f} \end{bmatrix} \times \nabla \left(\begin{bmatrix} n_b^0(x) & 0 \\ 0 & n_f^0(x) \end{bmatrix} \nabla \begin{bmatrix} \delta\mu_b(x) \\ \delta\mu_f(x) \end{bmatrix} \right). \quad (9)$$

In the outer fermionic shell, $\delta\mu_f^{\text{out}}$ satisfies an equation:

$$-m\omega^2 \delta\mu_f^{\text{out}} = \frac{\partial\mu_f^{\text{out}}}{\partial n_f} \nabla [n_f^{\text{out}}(x) \nabla \delta\mu_f^{\text{out}}]. \quad (10)$$

All modes can be classified by their parity with respect to the space inversion, $x \rightarrow -x$. Thus, we need to solve Eqs. (9) and (10) only in the region $x > 0$. For even modes, one has the following conditions at $x=0$:

$$\nabla \delta\mu_f(x=0) = 0, \quad \nabla \delta\mu_b(x=0) = 0. \quad (11)$$

For odd modes, analogous conditions are

$$\delta\mu_f(x=0) = 0, \quad \delta\mu_b(x=0) = 0. \quad (12)$$

Boundary conditions for fermions at the edge of the bosonic cloud, x_b , correspond to the continuity of $\delta\mu_f$ and v_f :

$$\nabla \delta\mu_f^{\text{out}}(x=x_b+0) = \nabla \delta\mu_f(x=x_b-0), \quad (13)$$

$$\delta\mu_f^{\text{out}}(x=x_b+0) = \delta\mu_f(x=x_b-0). \quad (14)$$

Two additional conditions come from the absence of the bosonic (fermionic) flow at $x_b(x_f)$:

$$n_b^0(x)v_b(x)|_{x \rightarrow x_b-0} = 0, \quad (15)$$

$$n_f^0(x)v_f(x)|_{x \rightarrow x_f-0} = 0. \quad (16)$$

Outside of the region of coexistence, the chemical potential and density of fermions are given by $\mu_f^{\text{out}} \sim (n_f^{\text{out}})^2$, $n_f^{\text{out}} \sim \sqrt{1-(x/x_f)^2}$, where x_f is the fermionic cloud size. For these functions, there exists a general nonzero solution of Eq. (10) that satisfies Eq. (16): $\delta\mu_f^{\text{out}} = \cos((\omega/\omega_0)\arccos x/x_f)$.

Substituting this into Eqs. (13) and (14), one has to solve eigenmode equations numerically for $x < x_b$, with five boundary conditions (13)–(15) and (11) or (12) depending on the parity. These boundary conditions are compatible, only if ω is an eigenfrequency. We choose to leave out condition (15), find a numerical solution to a system of differential equations, and check later if Eq. (15) is satisfied to identify the eigenfrequencies.

First, we apply this numerical procedure for weakly interacting regime, see the inset of Fig. 3. When $\gamma_0 \rightarrow 0$, boson and fermion clouds do not interact, and collective modes

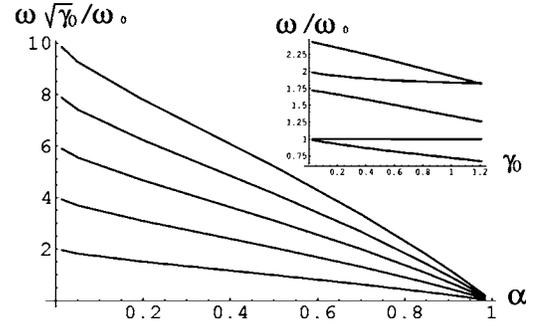


FIG. 3. Dependence of the energy of several lowest lying out of phase modes for $\gamma_0 \geq 1$ on overall boson fraction α , where γ_0 is the TG parameter in the center of a trap. Note, that we do not show in phase TG modes $\omega = n\omega_0$, since their frequency is much higher. The inset shows frequencies of low-lying modes in mean-field regime for an equal overall number of bosons and fermions.

coincide with purely bosonic or fermionic modes, with frequencies [23] $\omega^f = n\omega_0$ and $\omega^b = \omega_0 \sqrt{n(n+1)}/2$. As interactions get stronger, boson and fermion clouds get coupled, and all the modes except for dipole Kohn mode change their frequency. For the case of strongly interacting BF mixture, we find low-energy modes that correspond to the “out of phase” oscillations of bosons and fermions, and that keep the total density approximately constant. These modes can be understood as follows: For $\gamma \gg 1$, the energetic penalty for changing the relative density of bosons and fermions is small. The frequency of such relative modes scales as $\sim \omega_0/\sqrt{\gamma_0}$. In Fig. 3 we show the numerically obtained dependence of the energy of these low-energy modes as a function of the ratio of the total number of bosons and fermions.

Finally, we report the results of the calculations of the boson-boson and fermion-fermion correlation functions in systems with periodic boundary conditions. Such calculations can be performed in the regime $\gamma \gg 1$, due to “factorization” of the Bethe wave function, similar to Ref. [27]. The “spin” part of the wave function is related to the ground state of the spin XY model, and significant analytical progress can be made compared to the case of the spin-1/2 fermion model [27]. Details of these calculations will be presented elsewhere [28]. The Fourier transform of correlation

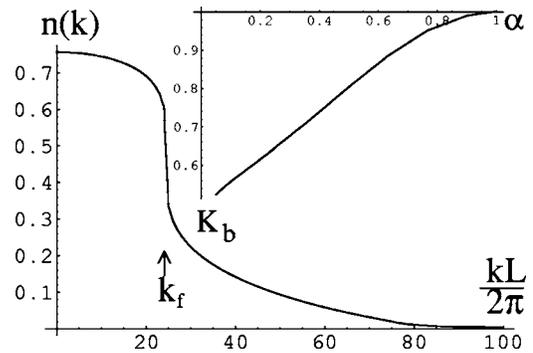


FIG. 4. Fourier transform of fermion-fermion correlation function ($N_b=51, N=100$)—The Fermi step gets strongly modified by interactions. The inset shows boson-boson correlation LL parameter K_b [defined by Eq. (17)] as a function of boson fraction α .

function is an occupation number $n(k)$, which can be measured directly in time-of-flight experiments [5] or using Bragg spectroscopy [29]. For fermions, $n(k)$ is shown in Fig 4. The discontinuity at $k_f = \pi n_f$ gets smeared out by interactions, but the overall change in $n(k)$ after crossing k_f remains quite large. For bosons, $n(k)$ is a monotonous decreasing function, with a singularity at $k=0$. Singularity at $k=0$ is governed by the long distance behavior of correlation function, characterized by the bosonic LL [30,31] parameter K_b :

$$\rho(0, \xi) \sim 1/|\xi|^{1/(2K_b)}, \quad n(k) \sim |k|^{-1+1/(2K_b)}, \quad (17)$$

where ξ is a distance between two points. For periodic boundary conditions, one can extract K_b using [31] a fit $\rho(0, \xi) \sim 1/|\sin(\pi\xi/L)|^{1/(2K_b)}$ for $\xi \gg L/N_b$. Results of the numerically extracted value of K_b for strong interactions are

shown in Fig. 4: Depending on the relative number of bosons and fermions, K_b changes from 0.5 to 1. We note that in the absence of fermions K_b changes from ∞ to 1 for interacting bosons as the interactions get stronger. In the presence of fermions, bosonic correlations decay faster than in the purely bosonic system. In principle, the low-energy physics of a BF mixture is characterized not only by boson-boson or fermion-fermion correlations, but also by the long-range behavior of BF correlations [12,14]. Our method can be applied to the evaluation of BF correlations and can be generalized to nonzero temperatures.

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