BOSE–EINSTEIN CONDENSATES

Fluctuating fringes

When two one-dimensional Bose–Einstein condensates interfere, they exhibit a fluctuating interference pattern. The full statistical distribution of the interference amplitude can be predicted, thanks to a remarkable connection to several exactly solvable problems.

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An interference experiment is perhaps the best way to demonstrate the importance of the concept of 'quantum phase'. When confronted with the double-slit experiment, we all have to accept quantum mechanics, however strange the theory may seem. Interference between Bose–Einstein condensates was first demonstrated[1] in 1996, providing clear evidence for phase coherence over the condensate. This finding also introduced a way to study quantum phases in many-particle systems. Interference between two one-dimensional Bose–Einstein condensates might seem uninteresting, as strong fluctuations in one dimension kill the global phase coherence. However, the interference between fluctuating local quantum phases still yields a non-vanishing interference fringe, whose amplitude also fluctuates. Thus the statistics of the interference fringe amplitude do indeed contain valuable information on the phase fluctuations. As reported on page 705 of this issue[2], Gritsev *et al*. predict exactly the full statistical distribution of the interference signal in the low-energy limit, using an interesting connection to many different physical problems in one dimension.

The strong fluctuations in one dimension make it difficult to apply conventional theoretical tools such as mean-field theory. On the other hand, the long quest to solve interacting quantum many-particle systems has led to several beautiful exact solutions specific to one dimension. As early as 1931, Bethe found exact eigenstates for the spin-1/2 Heisenberg chain[3]. The 'Bethe Ansatz' solution evolved into a rich field of integrable systems. In 1950, Tomonaga[4] showed that a certain model of interacting fermions in one dimension can be mapped exactly to a quantized sound wave: a system of non-interacting phonons. Later works by Luttinger, Luther, Haldane and many others have established that the quantized sound wave exactly describes the low-energy limit of a wide range of one-dimensional systems, including interacting fermions, spin chains and condensates of interacting bosons. Such a state is now called a Tomonaga–Luttinger liquid (TLL) or simply Luttinger liquid.

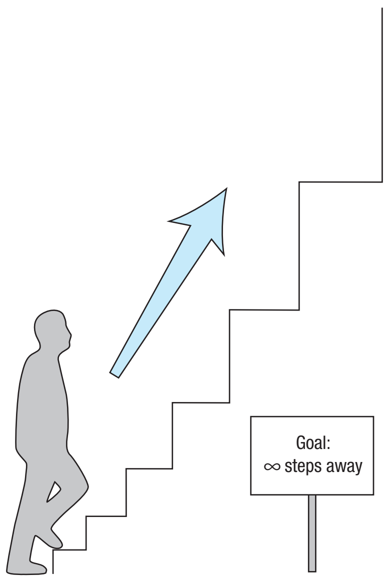

Figure 1 Too hard to climb? If each step represents the calculation of a moment, it seems impossible to determine the statistical distribution of the interference amplitude by computing all the moments. Higher-order moments are related to correlations among a larger number of points, and are increasingly difficult to evaluate. To determine the full distribution, we need to know all the moments up to infinite order. This is similar — in fact essentially identical — to the difficulty in carrying out perturbation theory on an interacting system to higher orders.

It is not specific to one dimension that the sound wave appears as a collective excitation in an interacting many-particle system. The miracle in one dimension is that all the fluctuations in the low energy limit can be captured in terms of the 'phonon field' describing the density fluctuation. This property is deeply connected to the fact that the 'phase field', which describes the local quantum phase fluctuation, can be defined in terms of the phonon field. These fields, which are two faces of a coin, satisfy a sort of uncertainty relation generalizing the well-known one between the particle number and the phase.

Interesting physical quantities, such as the local density of states, which is relevant for tunnelling and photoemission experiments, are often related to two-point correlation functions (the correlation function of operators at two different points). In a TLL, all such correlation functions decay algebraically as a function of distance, reflecting the critical nature of the state. This algebraic decay implies various power-law anomalies in physical quantities, some of which have indeed been confirmed experimentally.

Concerning the interference between two one-dimensional Bose–Einstein condensates, the interference signal is related to the correlation between local phases of the two condensates. If each condensate had a rigid global phase, the amplitude of the interference fringe would be constant, with just the relative phase being random. In fact, the phase also fluctuates within each one-dimensional condensate, leading to a statistical fluctuation of the amplitude of the interference fringe. Polkovnikov and co-workers[5] pointed out that the squared average of the amplitude is related to the two-point correlation function, which indicates the loss of phase coherence as a function of distance. However, the statistical distribution of the amplitude contains much more information than just its average value. In fact, the full distribution specifies all the moments, which are the average of $(2n)$th powers of the amplitude for all $n = 1, 2, 3, 4\ldots$ The $(2n)$th moment is related to the $2n$-point function that indicates the phase correlation at $2n$ different locations within each one-dimensional condensate. Thus the full statistical distribution of the interference amplitude provides an intriguing way to characterize the phase fluctuation, beyond the two-point correlation function that is usually studied.

At the same time, it presents a challenging problem for theorists. It seems impossible to obtain the full distribution function by evaluating all the moments up to infinite order. Actually, this is similar to the general difficulty in dealing with an interaction. A standard approach is the perturbative expansion in powers of the interaction. In principle it would be possible to obtain an exact result by summing over all orders of the perturbative expansion, where the mth-order term is given by an m-point correlation function in the free theory. But needless to say, this is usually practically impossible, with each successive step becoming more difficult (Fig. 1); otherwise many of the current issues would have been solved.

Amusingly, the above 'similarity' is not just an analogy. Let us consider the different physical problem of an impurity in a TLL, which is interesting in its own right[6]. The impurity tends to pin the phonon field at the impurity location to a preferred value. It induces a local anharmonicity, or interaction among phonons. The perturbative expansion with respect to this 'interaction' actually matches exactly the moments of the interference amplitude discussed above.

The remarkable point is that the impurity problem is exactly solvable[7] in the above context, using the Bethe Ansatz approach. Going backwards from the exact solution, we can read off all the moments for the interference amplitude distribution. Actually, to construct the full statistical distribution, it is necessary to solve an apparently unphysical problem, where the impurity strength is purely imaginary — a situation that cannot be easily handled using the original Bethe Ansatz approach. In 1971, Baxter introduced the concept of a Q-operator, in order to solve the spin-1/2 chain with a completely anisotropic exchange interaction[8]. Extended to conformal field theory, an important class of exactly solvable theory (which actually includes TLL as a special case) formulated in 1983 by Belavin, Polyakov and Zamolodchikov[9], the Q-operator turns out to give the desired solution for the imaginary impurity strength[2,10].

Connecting all these links, the full distribution function can finally be calculated[2]. Gritsev *et al.* have thus succeeded in making a highly non-trivial prediction of a quantum phenomenon, bringing together achievements of mathematical physics over several decades.

Our mathematically oriented colleagues often have had difficult confrontations with other physicists (sometimes including myself) who ask how all these funny mathematics are related to any physics. Those physicists might become more sympathetic if they recall their own difficulty with a non-scientist who questions the practical benefit of physics research. In parallel to the relation between science and technology, mathematics does not always follow demands, but could have quite unexpected applications in describing observable phenomena. As the work by Gritsev *et al.* testifies, this tendency seems even more true than ever; something difficult to find in nature could be engineered in artificial structures, especially with cold atoms. We can hope that cold atoms will link more (seemingly) 'purely imaginary' theories with reality.

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