Microscopic Electron Models with Exact SO(5) Symmetry

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We construct a class of microscopic electron models with exact SO(5) symmetry between antiferromagnetic and *d*-wave superconducting ground states. There is an exact one-to-one correspondence between both single-particle and collective excitations in both phases. SO(5) symmetry breaking terms can be introduced and classified according to irreducible representation of the exact SO(5) algebra. The resulting phase diagram and collective modes are identical to that of the SO(5) nonlinear σ model. [S0031-9007(98)05903-1]

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One of the most interesting features of the high-T_c superconductivity is the close proximity and the interplay between the antiferromagnetic (AF) and the *d*-wave superconducting (dSC) phases. Recently, a theoretical formalism was introduced based on the concept of a SO(5) symmetry between these two phases, and the resulting field-theoretical model describes the cuprate phase diagram and collective modes in a unified framework [1]. It was argued that the microscopic Hubbard model supports an approximate SO(5) symmetry [1–3].

In this paper, we construct a class of microscopic electron models with exact SO(5) symmetry. In this model, degeneracy between the AF and dSC phases can be demonstrated exactly, and both the fermionic single-particle and the bosonic collective modes can be mapped onto each other with a precise one-to-one correspondence. This model can be used as a starting point around which SO(5) symmetry-breaking interactions can be introduced and classified according to irreducible tensors of SO(5) algebra. It is shown that the resulting phase diagram and the collective modes are similar to those obtained from the effective SO(5) nonlinear σ model with anisotropic couplings [1,4]. The purpose of this paper is to demonstrate that the general SO(5) idea can be realized exactly by explicit microscopic Hamiltonians. The microscopic information extracted from this class of models, especially the behavior of the fermionic excitations across the AF/dSC transition, would greatly complement the effective field theory approach. Within this class of models, we have a consistent microscopic theory of the AF/dSC transition. Since both the AF and the dSC states are stable infrared fixed points, it is plausible that one can deform the parameters so that the microscopic SO(5) models can also serve as a paradigm for a much more general class of AF/dSC transitions, including those occurring in the high-T_c cuprates and 2D organics.

The first independent attempt to construct microscopic SO(5) invariant models was undertaken by Henley [5]. He independently made a crucial observation that, if one replaces the standard *d*-wave factor $\cos p_x - \cos p_y$ by $\mathrm{sgn}(\cos p_x - \cos p_y)$, the SO(5) algebra introduced in [1] closes exactly.

It is easy to write down many electron models with exact SU(2) spin rotation invariance, because the electron operator $c_{\mathbf{p}\sigma}$ forms a natural spinor representation of the SU(2) algebra. In writing down SU(2) invariant models, all we have to do is to contract the spinor indices in a natural way. Therefore, the first step towards constructing a SO(5) invariant electron model is to find a natural definition of a SO(5) spinor. Spinor representations of the SO(5) Lie algebra can be easily constructed using the Clifford algebra of five 4×4 Dirac matrices [6] satisfying $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$ $(a, b = 1, \ldots, 5)$, and the ten SO(5) rotation generators are given by $\Gamma^{ab} = -i[\Gamma^a, \Gamma^b]$. In this paper we shall use the following explicit representation for the Clifford algebra:

$$\Gamma^1 = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} \Gamma^{(2,3,4)} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & {}^t\vec{\sigma} \end{pmatrix}$$
$$\Gamma^5 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the usual 2×2 Pauli matrices, and the superscript t means transportation. We define a four-component spinor by

$$^t\Psi_{\mathbf{p}} = \{c_{\mathbf{p}\uparrow}, c_{\mathbf{p}\downarrow}, \phi_\pi(\mathbf{p}) c^\dagger_{-\mathbf{p}+\mathbf{Q},\uparrow}, \phi_\pi(\mathbf{p}) c^\dagger_{-\mathbf{p}+\mathbf{Q},\downarrow}\}, \quad (1)$$

where $\phi_\pi(\mathbf{p}) = \mathrm{sgn}(\cos p_x - \cos p_y) = \pm 1$, and $\mathbf{Q} = (\pi, \pi)$. Since we have two spin degrees of freedom at a given momentum $\mathbf{p}$, such a description must be redundant. Indeed, one can easily see that the spinors with momenta outside the magnetic Brillouin zone are related to the spinors inside the magnetic Brillouin zone by an "*R* conjugation"

$$\Psi_{\mathbf{p}+\mathbf{Q}} = \phi_\pi(\mathbf{p}) R \Psi^*_{-\mathbf{p}}. \quad (2)$$

The R matrix is an invariant tensor of the SO(5) algebra enjoying the following properties: $R\Gamma^a R = -{}^t\Gamma^a$, $R\Gamma^{ab} R = {}^t\Gamma^{ab}$. In our representation it takes the form $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. [The existence of such a matrix is related

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to the fact that the spinor representation of the SO(5) Lie algebra is pseudoreal. The σ_y matrix plays a similar role for SO(3)]. The $\Psi_{\mathbf{p}\alpha}$ spinors obey the anticommutation relations;

$$\{\Psi^\dagger_{\mathbf{p}\alpha}, \Psi_{\mathbf{p}'\beta}\} = \delta_{\alpha\beta}\delta_{\mathbf{p},\mathbf{p}'}\,, \tag{3}$$

$$\{\Psi^\dagger_{\mathbf{p}\alpha}, \Psi_{\mathbf{p}'\beta}\} = \{\Psi_{\mathbf{p}\alpha}, \Psi_{\mathbf{p}'\beta}\} = -\phi_\pi(\mathbf{p})R_{\alpha\beta}\delta_{\mathbf{p}+\mathbf{p}',\mathbf{Q}}\,. \tag{4}$$

If we restrict both $\mathbf{p}$ and $\mathbf{p}'$ inside the magnetic Brillouin zone, the right-hand side of the second equation vanishes and the $\Psi_{\mathbf{p}\alpha}$ spinors commute in the same way as the $c_{\mathbf{p}\sigma}$ spinors. They an be used to construct the SO(5) vector order parameter and the symmetry generators: $n_a = \frac{1}{4}\sum_{\mathbf{p}} w_{\mathbf{p}}\Psi^\dagger_{\mathbf{p}}\Gamma^a\Psi_{\mathbf{p}+\mathbf{Q}}$ and $L_{ab} = \frac{1}{8}\sum_{\mathbf{p}} \Psi^\dagger_{\mathbf{p}}\Gamma^{ab}\Psi_{\mathbf{p}}$. Here, $w_{\mathbf{p}} = w_{-\mathbf{p}}$. Note that the definition of the π operators ($L_{1(2,3,4)}, L_{(2,3,4)5}$) differs from the ones used in previous works [1–3], where they are electron pair operators on the nearest-neighbor (nn) sites. The problem with this kind of definition is that the commutator algebra does not close, and generates longer-ranged bonds. Naively, the condition for the closure of the SO(5) algebra appears to be overconstrained. The present paper and Ref. [5] start with electron pair operators with a long-ranged profile, given in real space by the lattice Fourier transform of $\sigma_\pi(\mathbf{p})$,

$$\sigma_\pi(m,n) = \frac{2}{\pi^2}\,\frac{1-(-)^{m+n}}{m^2-n^2}\,, \tag{5}$$

where $\mathbf{R} = (m,n)$ is a lattice point. It is truly remarkable that this simple choice closes the algebra exactly. Notice, that, while the π operators have long-ranged profiles, the dSC order parameter can still be short ranged with suitable choices of $w_{\mathbf{p}}$. Under the SO(5) rotations generated by the L_{ab}, $\Psi_{\mathbf{p}}$ transforms as a proper SO(5) spinor

$$[L_{ab}, \Psi_{\mathbf{p}\alpha}] = -\frac{1}{4}\,(\Gamma^{ab})_{\alpha\beta}\Psi_{\mathbf{p}\beta} \tag{6}$$

for all values of $\mathbf{p}$. Using these spinors, exact SO(5) invariant Hamiltonians can be constructed simply by proper contraction of the spinor indices.

We start with the kinetic term, and write it as

$$H_{\rm kin} = \sum_{\mathbf{p},\sigma} \varepsilon_{\mathbf{p}} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma} = \frac{1}{2}\sum_{\mathbf{p}} \varepsilon_{\mathbf{p}}\Psi^\dagger_{\mathbf{p}}\Psi_{\mathbf{p}}\,. \tag{7}$$

We see that the property $\varepsilon_{\mathbf{p}+\mathbf{Q}} = -\varepsilon_{\mathbf{p}}$, valid for a nn tight binding model, is crucial for this construction to work. In order to construct four-fermion interactions, we first note that a SO(5) spinor bilinear can in general be decomposed into a direct sum of a scalar, a vector, and an antisymmetric tensor, i.e., $4\times 4 = 1 + 5 + 10$. Therefore, general SO(5) invariant four-fermion interactions can be expressed as

$$\begin{aligned} H_{\rm int} = &\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} V_1(\mathbf{p},\mathbf{p}';\mathbf{q})\,(\Psi^\dagger_{\mathbf{p}}\Gamma^a\Psi_{\mathbf{p}+\mathbf{q}})\,(\Psi^\dagger_{\mathbf{p}'}\Gamma^a\Psi_{\mathbf{p}'-\mathbf{q}}) \\ &+ \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} V_2(\mathbf{p},\mathbf{p}';\mathbf{q})\,(\Psi^\dagger_{\mathbf{p}}\Gamma^{ab}\Psi_{\mathbf{p}+\mathbf{q}})\,(\Psi^\dagger_{\mathbf{p}'}\Gamma^{ab}\Psi_{\mathbf{p}'-\mathbf{q}}) \\ &+ \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} V_0(\mathbf{p},\mathbf{p}';\mathbf{q})\,(\Psi^\dagger_{\mathbf{p}}\Psi_{\mathbf{p}+\mathbf{q}})\,(\Psi^\dagger_{\mathbf{p}'}\Psi_{\mathbf{p}'-\mathbf{q}})\,. \end{aligned} \tag{8}$$

Since $L_{ab}(\mathbf{p},\mathbf{q}) \equiv \Psi^\dagger_{\mathbf{p}}\Gamma^{ab}\Psi_{\mathbf{p}+\mathbf{q}}$, $n_a(\mathbf{p},\mathbf{q}) \equiv \Psi^\dagger_{\mathbf{p}}\Gamma^a\Psi_{\mathbf{p}+\mathbf{Q}+\mathbf{q}}$ and $\rho(\mathbf{p},\mathbf{q}) \equiv \Psi^\dagger_{\mathbf{p}}\Psi_{\mathbf{p}+\mathbf{q}}$ are the true SO(5) tensor, vector, and scalar, respectively, for any $\mathbf{p}$ and $\mathbf{q}$; their inner products naturally gives a manifestly SO(5) invariant Hamiltonian.

Among three terms in $H_{\rm int}$, we concentrate on the vector interaction (first term) in all subsequent analysis, and assume a factorizable form $V_1(\mathbf{p},\mathbf{p}';\mathbf{q}) = -V_1(\mathbf{q})w_{\mathbf{p}}w_{\mathbf{p}'}$. This form is not necessary, but simplifies calculations. In real space,

$$\begin{aligned} H_{\rm int,1} = &-4\sum_{\ell,n} V_1(\mathbf{R}_\ell - \mathbf{R}_n)e^{i\mathbf{Q}\cdot(\mathbf{R}_\ell-\mathbf{R}_n)} \\ &\times\left[\mathbf{m}_\ell\cdot\mathbf{m}_n + \frac{1}{2}\,(\Delta_\ell\Delta^\dagger_n + \Delta^\dagger_\ell\Delta_n)\right]. \end{aligned} \tag{9}$$

Here, $\mathbf{m}_\ell$ and Δ_ℓ are Néel and d-wave pairing order parameters (operators) at site $\ell \equiv \mathbf{R}_\ell$, but with extended internal structures determined by $w_{\mathbf{p}}$. For the simplest choice $w_{\mathbf{p}} = 1$, they become

$$\mathbf{m}_\ell = \frac{1}{2}\,(\psi^\dagger_\ell\boldsymbol{\sigma}\psi_\ell - \chi^\dagger_\ell\boldsymbol{\sigma}\chi_\ell)e^{i\mathbf{Q}\cdot\mathbf{R}_\ell}, \tag{10}$$

$$\Delta^\dagger_\ell = \sum_j \phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j)\,(c^\dagger_{\ell\uparrow}c^\dagger_{j\downarrow} - c^\dagger_{\ell\downarrow}c^\dagger_{j\uparrow})\,. \tag{11}$$

Here, we introduced two-component spinors $\psi_\ell = {}^t(c_{\ell\uparrow}, c_{\ell\downarrow})$ and $\chi_\ell = (-e^{i\mathbf{Q}\cdot\mathbf{R}_\ell})\times {}^t(b_{\ell\uparrow}, b_{\ell\downarrow})$ with $b_{\ell\sigma} = \sum_j \phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j)c_{j\sigma}$. The pair wave function for dSC condensate is described by ϕ_π and is long ranged. For the choice $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$, we obtain

$$\mathbf{m}_\ell = \frac{e^{i\mathbf{Q}\cdot\mathbf{R}}}{2}\sum_i \phi_M(\mathbf{R}_\ell - \mathbf{R}_i)\,(\psi^\dagger_i\boldsymbol{\sigma}\psi_\ell - \chi_i^\dagger\boldsymbol{\sigma}\chi_\ell)\,, \tag{12}$$

$$\Delta^\dagger_\ell = \sum_{i,j} \sigma_M(\mathbf{R}_\ell - \mathbf{R}_i)\phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j)\,(c^\dagger_{i\uparrow}c^\dagger_{j\downarrow} - c^\dagger_{i\downarrow}c^\dagger_{j\uparrow})\,, \tag{13}$$

where

$$\phi_M(m,n) = \frac{4}{\pi^2}\,\frac{1+(-)^{m+n}}{[(m+n)^2-1][(m-n)^2-1]}\,. \tag{14}$$

The interaction between the centers of mass of $\mathbf{m}$ or Δ fields is controlled by $V_1(\mathbf{R})$. If we take $V_1(\mathbf{q})$ to be a δ function at $\mathbf{q} = \mathbf{Q}$, the Δ part in $H_{\rm int,1}$ becomes the usual BCS reduced Hamiltonian for nn d-wave pairing. If

$V_1(\mathbf{q})$ is taken to be a Lorenzian around $\mathbf{q} = \mathbf{Q}$, the real space form of the spin interaction resembles the potential induced by the AF paramagnon exchange [7–9].

It is not difficult to find degeneracy between AF and dSC states in the usual treatment of mean field theories. However, their excitation spectra are generally different, and quantum fluctuations may remove this degeneracy. In the SO(5) invariant models, symmetry not only ensures exact degeneracy of the ground states, but also ensures exact one-to-one correspondence between their excitation spectra. This fact is formulated as follows:

Theorem 1.—If $|\Psi_0\rangle$ is a ground state of a SO(5) invariant Hamiltonian with AF broken symmetry (say, in the n_2 direction), i.e., $\langle\Psi_0|n_a|\Psi_0\rangle = \delta_{2,a}A$, then $|\Psi_0'\rangle = e^{i(\pi/2)L_{12}}|\Psi_0\rangle$ is a degenerate ground state with dSC broken symmetry (in n_1 direction), i.e., $\langle\Psi_0'|n_a|\Psi_0'\rangle = \delta_{1,a}A$. Furthermore, all excited states of the AF ground state can be mapped to excited states of the dSC ground state at the same energy by the $e^{i(\pi/2)L_{12}}$ operator.

The proof of this theorem is elementary, since L_{12} commutes with the Hamiltonian, and $e^{-i(\pi/2)L_{12}}n_1e^{i(\pi/2)L_{12}} = n_2$. In the following, we shall illustrate this powerful theorem in an explicit mean field calculation. We take a "generalized BCS reduced Hamiltonian" by selecting $V_1(\mathbf{q}) = V_1\delta_{\mathbf{q},\mathbf{Q}}$ in the vector interaction. The Green's function in the presence of a mean field $\langle n^a_{\mathbf{p}}\rangle = \frac{1}{4}\langle\Psi^\dagger_{\mathbf{p}}\Gamma^a\Psi_{\mathbf{p}+\mathbf{Q}}\rangle$ is given by

$$G_{\alpha\beta}(\mathbf{p},\mathbf{p}';\omega) = -i\int dt e^{i\omega t}\langle T\Psi_{\mathbf{p},\alpha}(t)\Psi^\dagger_{\mathbf{p}',\beta}(0)\rangle = \frac{(\omega+\varepsilon_{\mathbf{p}})\delta_{\alpha\beta}\delta_{\mathbf{p},\mathbf{p}'} + \Delta^a_{\mathbf{p}}\Gamma^a_{\alpha\beta}\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2-\varepsilon^2_{\mathbf{p}}-(\Delta^a_{\mathbf{p}})^2+i\delta}, \tag{15}$$

where $\Delta^a_{\mathbf{p}} = -16V_1w_{\mathbf{p}}\sum_{\mathbf{k}} w_{\mathbf{k}}\langle n^a_{\mathbf{k}}\rangle$. This manifestly SO(5) invariant Green's function shows explicitly that the AF quasiparticles can be mapped onto dSC quasiparticles. In particular, the AF Green's function (in the n_2 direction) can be obtained directly from the dSC Green's function (in the n_1 direction) by a simple rotation: $G^{\mathrm{AF}} = e^{-i(\pi/2)\Gamma_{12}}G^{\mathrm{SC}}e^{i(\pi/2)\Gamma_{12}}$. If we take $w_{\mathbf{p}} = 1$, the AF quasiparticles have a full s-wave gap, while the dSC quasiparticles have a full d-wave gap, with step discontinuity at $(\pm\pi/2,\pm\pi/2)$ points. For the choice of $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$, the dSC quasiparticles have the usual $\cos p_x - \cos p_y$ gap behavior, while the AF quasiparticles have an anisotropic s-wave gap with nodes at $(\pm\pi/2,\pm\pi/2)$ points (Fig. 1). Because the AF nodes are not "topological," any small interactions will remove it [10]. In either case, the amplitude of the gaps is the same in both phases.

As symmetry-breaking perturbations to the above SO(5) invariant Hamiltonian, we consider two typical terms. One is the coupling to external fields B_{ab},

$$H_{\mathrm{ext}} = -\sum_{a<b} B_{ab}L_{ab}\,. \tag{16}$$

A particular example of this field is the chemical potential

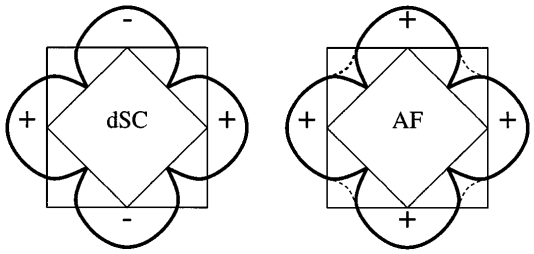

FIG. 1. The superconducting (dSC) and antiferromagnetic (AF) gaps. The solid (dotted) line in the AF gap is for the case with (without) SO(5) symmetry.

$B_{15} = -2\mu$, leading to $H_\mu = 2\mu L_{15}$. The other is the anisotropy energy

$$H_g = -\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\sum_{a=2,3,4} g(\mathbf{q})(\Psi^\dagger_{\mathbf{p}}\Gamma^a\Psi_{\mathbf{p}+\mathbf{q}}) \times (\Psi^\dagger_{\mathbf{p}'}\Gamma^a\Psi_{\mathbf{p}'-\mathbf{q}}) \tag{17}$$

between AF and dSC states. To study the spectrum of $H = H_{\mathrm{kin}} + H_{\mathrm{int},1} + H_\mu + H_g$, we take $g(\mathbf{q}) = g\delta_{\mathbf{q},\mathbf{Q}}$ and again use mean field approximation. In the dSC phase, $\langle n^a_{\mathbf{p}}\rangle$ lies in the plane (n_1, n_5). We choose it in the n_1 direction. Then the Green's function is given by

$$G^{\mathrm{SC}}(\mathbf{p},\mathbf{p}',\omega) = \begin{pmatrix} \frac{(\omega+\varepsilon_{\mathbf{p}}-\mu)1\delta_{\mathbf{p},\mathbf{p}'}}{\omega^2-(\varepsilon_{\mathbf{p}}-\mu)^2-\Delta^2_{\mathbf{p}}+i\delta} & \frac{-i\sigma_y\Delta_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2-(\varepsilon_{\mathbf{p}}+\mu)^2-\Delta^2_{\mathbf{p}}+i\delta} \\ \frac{i\sigma_y\Delta_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2-(\varepsilon_{\mathbf{p}}-\mu)^2-\Delta^2_{\mathbf{p}}+i\delta} & \frac{(\omega+\varepsilon_{\mathbf{p}}+\mu)1\delta_{\mathbf{p},\mathbf{p}'}}{\omega^2-(\varepsilon_{\mathbf{p}}+\mu)^2-\Delta^2_{\mathbf{p}}+i\delta} \end{pmatrix},$$

where $\Delta_{\mathbf{p}} = -16V_1\phi_\pi(\mathbf{p})w_{\mathbf{p}}\sum_{\mathbf{k}} w_{\mathbf{k}}\langle n^1_{\mathbf{k}}\rangle \equiv \Delta_0\phi_\pi(\mathbf{p})w_{\mathbf{p}}$ and the g term drops out because of symmetry mismatch. Δ_0 is determined by the gap equation $1 = 16V_1\sum_{\mathbf{k}}\frac{w^2_{\mathbf{k}}}{2E_{\mathbf{k}}}$, where $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}}-\mu)^2+\Delta^2_{\mathbf{k}}}$. For the choice $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$, we have a usual d-wave gap. In the AF phase, $\langle n^a_{\mathbf{p}}\rangle$ lies in the (n_2, n_3, n_4) space. If we pick the n_4 direction, we have

$$G^{\mathrm{AF}}(\mathbf{p},\mathbf{p}',\omega) = \begin{pmatrix} \frac{(\omega_+ +\varepsilon_{\mathbf{p}})1\delta_{\mathbf{p},\mathbf{p}'}+\Delta_{\mathbf{p}}\sigma_z\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2_+-\varepsilon^2_{\mathbf{p}}-\Delta^2_{\mathbf{p}}+i\delta} & 0 \\ 0 & \frac{(\omega_- +\varepsilon_{\mathbf{p}})1\delta_{\mathbf{p},\mathbf{p}'}+\Delta_{\mathbf{p}}\sigma_z\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2_--\varepsilon^2_{\mathbf{p}}-\Delta^2_{\mathbf{p}}+i\delta} \end{pmatrix}.$$

Here $\omega_\pm = \omega \pm \mu$ and $\Delta_{\mathbf{p}} = -16\sum_{\mathbf{k}}(V_1w_{\mathbf{p}}w_{\mathbf{k}} + g)\langle n^4_{\mathbf{k}}\rangle \equiv w_{\mathbf{p}}\Delta_0 + \Delta_g$. Δ_0 and Δ_g are determined by the gap equation $\Delta_{\mathbf{p}} = 16\sum_{\mathbf{k}}(V_1w_{\mathbf{p}}w_{\mathbf{k}} + g)\frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}$. The real space form of $\Delta_{\mathbf{p}}$ has an on-site contribution from Δ_g and a long-ranged contribution proportional to $\Delta_0\phi_M(\mathbf{R})$. We see that the g terms leave the dSC gap unaffected, while it removes the AF gap node (See Fig. 1). The ground state energy curves are shown in Fig. 2. The "superspin flop" transition from AF to dSC occurs at $\mu = 0$ for $H_g = 0$, while it occurs at a finite value of μ_c for $g \neq 0$. In this case, the AF/dSC transition is first order, with a finite jump in hole density x_c (See Fig. 2).

While the above pictures are based on the mean field approximation, some exact statements can be made about the AF/dSC transitions. SO(5) is a rank 2 algebra, and we can choose $Q = -L_{15}$ and $S_z = -L_{23}$ as the members

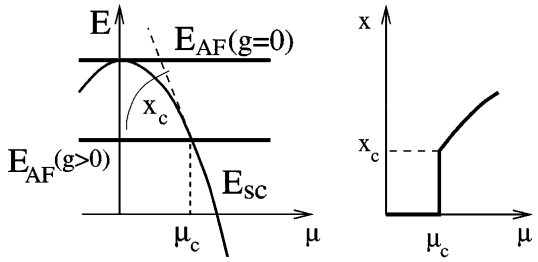

FIG. 2. The ground state energy G in both AF and SC phases as functions of μ (left) and the electron density x versus μ in the presence of anisotropy energy H_g (right).

of the Cartan (maximal commutative) subalgebra. In addition, we have the Casimir operator $C = \sum_{a<b} L_{ab}^2$, which commutes with all of the generators and has an eigenvalue $l(l + 3)$. The set (Q, S_z, C) forms a Cartesian coordinate system labeling the quantum numbers of all states in the Hilbert space. If we consider only states with even number of electrons, these states form a pyramid, with the $l = 0$ singlet on top, the $l = 1$ vector next, and the $l = 2$ traceless symmetric tensor on the third layer, etc. States on the same layer are all connected by the repeated actions of the eight root generators.

Theorem 2.—In SO(5) invariant models, it is sufficient to diagonalize the Hamiltonian at half-filling with $Q = 0$ and $S_z = 0$. All of the other states (with even number of electrons) in the Hilbert space can be obtained from these states through the action of $(S_x, S_y, \boldsymbol{\pi}, \boldsymbol{\pi}^\dagger)$.

In this sense, states at half-filling fully determine the states away from half-filling. In the presence of the H_μ term, the $\pi_\alpha^\dagger$ and π_α operators are exact eigenoperators of the Hamiltonian with eigenvalue $\pm 2\mu$. Therefore, H_μ commutes with the Casimir operator, and simply shifts the energy of the $Q \neq 0$ states linearly without changing the wave function of these states. In a system with spontaneous symmetry breaking, lowest states with different l quantum numbers are separated by an inverse system size. In the infinite system limit, these shifts, due to chemical potential, converge to the parabola as depicted in Fig. 2 [11].

The symmetry-breaking terms, H_g and H_μ, produce the mass gap in the Goldstone mode spectrum. For $H_g = 0$, the mass of the π triplet mode is exactly $2|\mu|$. For finite H_g we employed the equations of motion (EOM) [12,13] for n^a to get the spectrum of collective modes. In the dSC state, we take $\langle n_a(\mathbf{q})\rangle = \langle n_1\rangle \delta_{a,1}\delta_{\mathbf{q},\mathbf{0}}$, and linearize the EOM to obtain the energy of the π triplet $\omega_0 = 2\sqrt{\mu^2 - \mu_c^2}$, where $\mu_c = \langle n_1\rangle \sqrt{gV_1}$, $g \equiv g(\mathbf{Q})$, and $V_1 \equiv V_1(\mathbf{Q})$, which is also consistent with the result of [1]. Similar calculation in the AF state gives the energies of the π doublet $\omega_0 = 2\langle n_4\rangle \sqrt{g(V_1 + g)} \pm 2\mu$, where we assumed AF ordering along n_4. We therefore see that the two symmetry-breaking terms g and μ partially compensate each other for the π triplet and $Q = -2$ π doublet.

In conclusion, we have constructed a class of electron models with exact SO(5) symmetry. If the model has an AF ground state at half-filling, it will have a dSC ground state away from half-filling. The phase diagram and the collective mode spectrum of these models are similar to the real high-T_c materials. However, our results also show that a significant SO(5) symmetry-breaking term H_g is required to produce the asymmetry in the size and angular distribution of the insulating and superconducting gaps. It is encouraging that this symmetry-breaking term transforms as a simple irreducible tensor under SO(5), and its consequences can be worked out systematically by the Wigner-Eckart theorem. Recent numerical calculations by Eder, Hanke, and Zhang [14] show that the two symmetry-breaking terms, H_g and H_μ, have compensating effects, so that a approximate SO(5) multiplet structure can be observed in the hole part of the spectrum in the t-J model.

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