

Many-body protected entanglement generation in interacting spin systems

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We discuss a method to achieve decoherence resistant entanglement generation in two level spin systems governed by gapped and multi-degenerate Hamiltonians. In such systems, while the large number of degrees of freedoms in the ground state levels allows to create various quantum superposition, the energy gap prevents decoherence as local excitations become energetically suppressed. We apply the protected evolution to achieve decoherence resistant generation of many particle GHZ states and show it can significantly increase the sensitivity in frequency spectroscopy. We discuss how to engineer the desired many-body protected manifold in two specific physical systems, trapped ions and neutral atoms in optical lattices, and present simple expressions for the fidelity of GHZ generation under non-ideal conditions.

I. INTRODUCTION

It is well-known that entangled atomic states (e.g. so-called spin squeezed states) potentially allow to significantly improve resolution in Ramsey spectroscopy [1, 2]. Entangled states are also a fundamental resource in quantum information and quantum computation science [3, 4]. However, in practice entangled states are difficult to prepare and maintain as noise and decoherence rapidly collapses them into classical statistical mixtures. Thus one of the most important challenges in modern quantum physics is the design of robust and most importantly decoherence resistant methods for entanglement generation.

We have recently proposed a method [5] that allows for noise resistant generation of entangled states. The method uses the energy gap of properly designed gapped-multi-degenerate Hamiltonians. While the large number of degrees of freedoms in the ground state manifold of such systems allows to create various quantum superposition and to exploit rich dynamical evolution (suitable for example for precision spectroscopy), the energy gap prevents decoherence as local excitations become energetically suppressed. A simple example of a many-body spin Hamiltonian that illustrates the idea of our scheme is a multi-spin system with isotropic ferromagnetic interactions. These interactions will naturally align the spins. While all of the spins can be rotated together around an arbitrary axis without cost of energy, local spin flips are energetically forbidden.

This paper presents a detailed analysis of this method when applied to trapped ions and cold atoms in optical lattices. We demonstrate its applicability for decoherence resistant generation of N -particle Greenberger-Horne-Zeilinger (GHZ) states [6] and the potential of the latter to be used for Heisenberg-limited spectroscopy. The paper is organized as follows: In Sec II we review one of the standard procedures used to generate multi-particle GHZ entangled states in ion traps and demonstrate the detrimental effect of phase decoherence in such entanglement generation schemes. In Sec III we explain the idea of a many-body protected manifold (MPM) and discuss how and under what conditions it can significantly reduce the effect of decoherence. In Sec. IV we show the applicability of the gap protected evolution to precision measure-

ments and discuss the significant improvement in phase sensitivity that it might provide. In section V we elaborate on the physical resources required for the implementation of the gap protected Hamiltonian in trapped ions and discuss the advantages and disadvantages of its implementation with respect to standard unprotected Hamiltonians. In Sec. VI we study how to engineer the long range interactions required for the gap protected evolution in optical lattice systems interacting via short range interactions and discuss the effectiveness of the MPM for GHZ generation. In Sec. VII we analyze in such systems the effect of non-ideal conditions such as the magnetic trapping confinement and finally conclude in Sec.VIII.

II. MULTI-PARTICLE ENTANGLEMENT GENERATION

A. Ideal Case

In this section we start by reviewing a method to generate multi-particle entangled states in a system of N spin 1/2 atoms by time evolution under the so called squeezing Hamiltonian.

$$\hat{H}_z = \chi \hat{J}_z^{(0)2}. \quad (1)$$

As shown in Ref. [7, 8] the \hat{H}_z Hamiltonian can be implemented in trapped ions systems by using the collective vibrational motion of the ions in a linear trap driven by illuminating them with a laser field. In actual experiments \hat{H}_z has been realized by means of rotations and phase gates [1, 2, 9]. In Eq.(1) we used $\hat{J}_\alpha^{(0)}$ to denote the collective spin operators of the N atoms: $\hat{J}_\alpha^{(0)} = \frac{1}{2} \sum_i \hat{\sigma}_i^\alpha$, where $\alpha = x, y, z$ and $\hat{\sigma}_i^\alpha$ is a Pauli operator acting on the i^{th} atom. Moreover we have identified the two relevant internal states of the atoms with the effective spin index $\sigma = \uparrow, \downarrow$. In this manuscript we will use units such that $\hbar = 1$ and assume N to be even.

An appropriate basis to describe the dynamics of the system is the one spanned by collective pseudo-spin states denoted as $|J, M, \beta\rangle_z$ [10]. These states satisfy the eigenvalue relations $\hat{J}^{(0)2}|J, M, \beta\rangle_z = J(J+1)|J, M, \beta\rangle_z$ and $\hat{J}_z^{(0)}|J, M, \beta\rangle_z = M|J, M, \beta\rangle_z$, with $J = N/2, \dots, 0$ and $-J \leq M \leq J$. β is an additional quantum number associ-

ated with the permutation group which is required to form a complete set of labels for all the 2^N possible states.

The entanglement generation process starts by preparing the system at $t = 0$ in a fully polarized state along the x direction, $|N/2, N/2\rangle_x$. As any state with $J = N/2$ is uniquely characterized by M , for denoting them we omit the additional β label. Fully polarized states along x can be written as a superposition of states with different M values along the z direction: $\sum_M C_M |N/2, M\rangle_z$. During the evolution the Hamiltonian imprints an M^2 dependent phase to the different components. As the system evolves, at first the winding of the phases leads to a collapse of $\langle \hat{J}_x^{(0)} \rangle$. However, at time $\chi t_{rev} = \pi$ all components rephase with opposite polarization, and a perfect revival of the initial state is observed with $\langle \hat{J}_x^{(0)} \rangle = -N/2$ (see Fig.1). Specifically, the time evolution of $\langle \hat{J}_x^{(0)} \rangle$ for systems with $N \gg 1$ can be shown to be given by:

$$\langle \hat{J}_x^{(0)} \rangle = \frac{N}{2} \sum_{k=0,1,2,\dots} (-1)^k e^{-N/2(\chi t - k\pi)^2} \quad (2)$$

Right at time $t_0 = t_{rev}/2$ the system becomes a macroscopic superposition of fully polarized states along the $\pm x$ direction, i.e. a N -particle GHZ state of the form

$$|\psi_x^{GHZ}\rangle \equiv \frac{1}{\sqrt{2}} \left(e^{-i\phi_+} |N/2, N/2\rangle_x + e^{i\phi_-} |N/2, -N/2\rangle_x \right), \quad (3)$$

with and ϕ_{\pm} real phases given by $-\pi/4$ and $\pi/4 + N\pi/2$.

Recent experiments [1, 2] have used this type of scheme to generate GHZ states in trapped ions with the aim to perform precision measurements of ω_0 , the energy splitting between \uparrow and \downarrow levels. Ideally the use of GHZ states should enhance the phase sensitivity to the fundamental Heisenberg limit [11]. However, decoherence significantly limited the applicability of the method and in practice even for only six ions, the achieved phase sensitivity was significantly below the fundamental limit.

B. Effect of decoherence

To understand the detrimental effect of decoherence we first assume that the dominant type of decoherence is single-particle dephasing. Such dephasing comes from processes that, while preserving the populations in the atomic levels, randomly change the phases leading to a decay of the off-diagonal density matrix elements. We model the phase decoherence by adding to Eq. (1) the following Hamiltonian [12]

$$\hat{H}_{env} = \frac{1}{2} \sum_i h_i(t) \hat{\sigma}_i^z, \quad (4)$$

where $h_i(t)$ are assumed to be independent stochastic Gaussian processes with zero mean and with autocorrelation function $\overline{h_i(t)h_j(\tau)} = \delta_{ij}f(t-\tau)$. Here the bar denotes av-

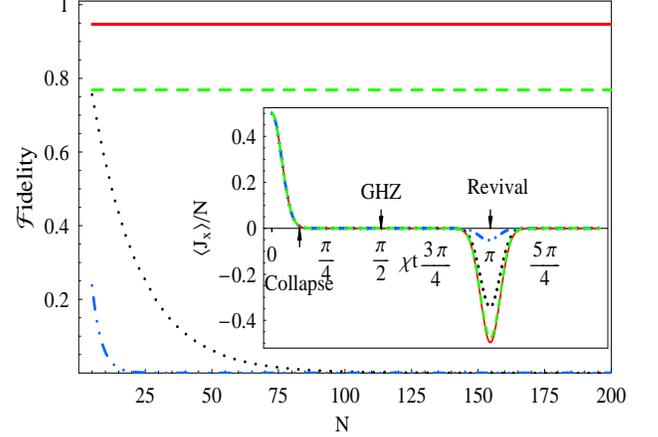


FIG. 1: (Color online) Fidelity of GHZ generation vs. N with and without gap protection. We assume a noise with step-like spectral density with amplitude f and cutoff frequency $\omega_c = \chi$. The dashed green and solid red lines are for a protected system, $\lambda = 50$, with $f = 0.6\chi$ and $f = 0.1\chi$ respectively. The dot-dashed blue and black dotted lines are for an unprotected system, $\lambda = 0$, and same decoherence parameters. In the inset we show $\langle \hat{J}_x^{(0)}(t) \rangle / N$ for $N = 50$.

eraging over the different random outcomes. In what follows we will use the property that zero mean Gaussian variables satisfy $\exp[-i \int_0^t d\tau h(\tau)] = \exp[-\Gamma(t)]$, with $\Gamma(t) = \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 f(t_1 - t_2)$ [13].

Phase decoherence causes an exponential decay of the revival peak and an N dependent decay of the fidelity, defined as $\mathcal{F}(t_0) = \langle \psi_x^{GHZ} | \hat{\rho}(t_0) | \psi_x^{GHZ} \rangle$ (See Fig. 1). Qualitatively the effect of phase decoherence on the evolving state can be understood from the energy levels of \hat{H}_z (See Fig. 2). In contrast to \hat{H}_z which commutes both with $\hat{J}_z^{(0)}$ and $\hat{J}^{(0)2}$, \hat{H}_{env} commutes with $\hat{J}_z^{(0)}$ but does not commute with $\hat{J}^{(0)2}$. Therefore, in the presence of phase decoherence transitions between different J subspaces are allowed as long as M is conserved. As all the states with the same $|M|$ value are degenerate, there is no energy barrier to prevent such transitions and very quickly the initially populated $J = N/2$ manifold is depleted and the fidelity of generating the GHZ state significantly degraded. Moreover, the degradation scales exponential with increasing N due to the exponential scalability of the number of accessible states to which the initially $J = N/2$ population can be transferred to.

Quantitatively the effect of decoherence can be calculated by using the uncoupled spin basis as it diagonalizes the total Hamiltonian. Each state in this basis can be label as $\{|n^{(k)}\rangle = |s_1^k, s_2^k, \dots, s_N^k\rangle\}$, where $s_i^k = \pm 1$ for $\uparrow\downarrow$ and $k = 1, \dots, 2^N$.

If at $t = 0$ the reduced density matrix of the system, $\hat{\rho}$, is given by $\hat{\rho} = \sum_{k,l} \rho_{k,l}(0) |n^{(k)}\rangle \langle n^{(l)}|$, after time t

$$\rho_{k,l}(t) = \rho_{k,l}(0) e^{\frac{i}{2} \chi t [(\sum_{i=1}^N s_i^k)^2 - (\sum_{i=1}^N s_i^l)^2]} \times e^{-i \sum_{i=1}^N \int_0^t d\tau h_i(\tau) (s_i^k - s_i^l)} \quad (5)$$

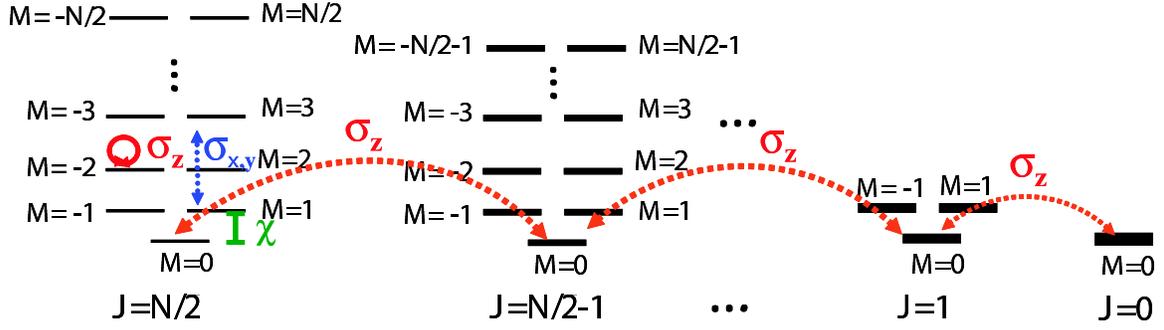


FIG. 2: Schematic representation of the energy levels of the $\chi \hat{J}_z^{(0)2}$ Hamiltonian and the effect of the different type of noise. As states with different J but equal $|M|$ are degenerate, in the presence of phase decoherence (σ_z noise) they are populated during the time evolution. $\sigma_{x,y}$ noises couple states which differ by ± 1 units of M and therefore the small energy gap between them, of the order of χ , naturally protects the system from these type of processes.

As a consequence, $\langle \hat{J}_x^{(0)}(t) \rangle = \overline{\text{Tr}[\hat{J}_x^{(0)} \hat{\rho}(t)]} = e^{-\Gamma(t)} \langle \hat{J}_x^{(0)}(t) \rangle|_{\Gamma=0}$ with $\langle \hat{J}_x^{(0)}(t) \rangle|_{\Gamma=0}$ the expectation value in the absence of noise (Eq.(2)). The factor $e^{-\Gamma(t)}$ comes from the fact that the operator $\hat{J}_x^{(0)} = \sum_i \hat{\sigma}_i^x$ only probes one-particle coherence, i.e it only connects states with exactly one spin flipped.

Assuming that at $t = 0$ all atoms are polarized in the x direction, i.e. $\rho_{k,l}(0) = 2^{-N}$, one can show from Eq. (5) that the fidelity is degraded to

$$\mathcal{F}(t_o) = \frac{1}{2^{2N}} \sum_{l,k} e^{-\frac{\Gamma(t_o)}{4} \sum_{i=1}^N (s_i^k - s_i^l)^2} = \left(\frac{1 + e^{-\Gamma(t_o)}}{2} \right)^N \quad (6)$$

III. PROTECTED DYNAMICS

A. Manybody protected manifold (MPM)

Let us now consider what happens if in addition of \hat{H}_z we assume that there is an isotropic infinite range ferromagnetic interactions between the spins so the system Hamiltonian is described by the Hamiltonian $\hat{H}_c = \hat{H}_{prot} + \hat{H}_z$, where

$$\hat{H}_{prot} = -\lambda \hat{J}^{(0)2} \quad (7)$$

Here we have assumed that all atoms are in different orbitals but have enough of spatial overlap that every spin interacts with every other spin.

The isotropic Hamiltonian \hat{H}_{prot} has a ground state manifold spanned by a set of $N + 1$ degenerate states. They lie on the surface of the Bloch sphere with maximal radius $J = N/2$ and are totally symmetric, i.e. invariant with respect to particle permutations. There is a finite energy gap $E_g = \lambda N$ that isolates the ground state manifold from the rest of the Hilbert space. This gap is the key for the many-body protection against decoherence. Hereunder we will refer to the ground state manifold as the *many-body protected manifold*

(MPM).

B. Protection against phase decoherence

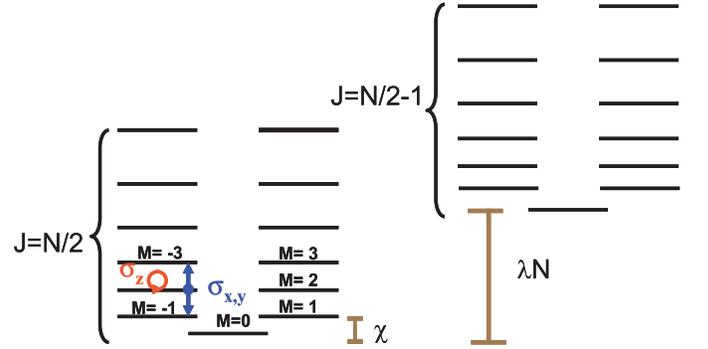


FIG. 3: Schematic representation of the energy levels of the $\hat{H}_{prot} + \chi \hat{J}_z^{(0)2}$ Hamiltonian. \hat{H}_{prot} lifts the degeneracy of the different J manifolds and suppresses (in the slow noise limit) couplings between them. So if a $t = 0$ the system is in the MPM it remains there.

The low energy spectrum of \hat{H}_c is shown in Fig. 3. As \hat{H}_z commutes with \hat{H}_{prot} , in the absence of decoherence the latter does not affect at all the GHZ generation dynamics, however in the presence of decoherence the latter does significantly reduces the effect of local environmental noise. The protection can be best understood by using the basis of collective states. In terms of collective spin operators \hat{H}_{env} can be written as:

$$\hat{H}_{env} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} g^k(t) \hat{J}_z^{(k)}, \quad (8)$$

where $g^k(t) = \frac{1}{\sqrt{N}} \sum h_j(t) e^{-i \frac{2\pi j k}{N}}$ and $\hat{J}_z^{(k)} = \frac{1}{2} \sum \hat{\sigma}_j e^{i \frac{2\pi j k}{N}}$. Note that allowed transitions must conserve

M as both the system and noise Hamiltonian commute with $\hat{J}_z^{(0)}$. In the presence of a large energy gap E_g , one can distinguish two different type of processes: (i) Decoherence effects that take place within the MPM due to the collective dynamics induced by the $k = 0$ component of \hat{H}_{env} , and (ii) transitions across the gap induced by the inhomogeneous terms. The later couple the MPM with the rest of the system, however they are nonenergy conserving process and consequently perturbatively weak.

Using a perturbative analysis, and assuming that at $t = 0$ the system lies within the MPM, the evolution of the projection of the density matrix on the MPM: $\rho_{M\tilde{M}} \equiv {}_z\langle N/2, \tilde{M} | \rho | N/2, M \rangle_z$ can be written as

$$\rho_{M,\tilde{M}}(t) = \rho_{M,\tilde{M}}(0) e^{it\chi(M^2 - \tilde{M}^2)} e^{i(\theta_M - \theta_{\tilde{M}})} e^{-\frac{1}{2}(\gamma_M - \gamma_{\tilde{M}})}, \quad (9)$$

Here

$$\theta_M(t) \equiv \langle \frac{N}{2}, M | \int_0^t d\tau \hat{H}_{env}(\tau) | \frac{N}{2}, M \rangle = \frac{M}{\sqrt{N}} \int_0^t g^0(\tau), \quad (10)$$

accounts for the dynamics induced by the noise within the MPM and

$$\gamma^M(t) = \sum_{J \neq N/2, \beta} \left| \int_0^t d\tau \mathcal{M}_{J,\beta}^M e^{i\tau\omega_{J,\beta}} \right|^2, \quad (11)$$

takes into account the depletion of the $J = N/2$ levels due to transition matrix elements between $|\frac{N}{2}, M\rangle_z$ and states outside the MPM: $\mathcal{M}_{J,\beta}^M = {}_z\langle \frac{N}{2}, M | \hat{H}_{env} | J, M, \beta \rangle_z$. $\omega_{J,\beta}$ are the respective energy splittings. Because \hat{H}_{env} is a vector operator, according to the Wigner-Eckart theorem, \hat{H}_{env} only couples the states in the MPM with states which have $J = N/2 - 1$ and thus with excitation energy λN .

Assuming the power spectrum of the noise, $f(\omega) \equiv \int dt e^{-i\omega t} f(t)$, to have a cut-off frequency ω_c (e.g. $f(\omega) = f$ for $\omega \leq \omega_c$ and 0 otherwise), we find that

$$\overline{\gamma^M(t)} \approx \frac{N^2 - 4M^2}{N} f \int_0^{\omega_c} d\omega \left(\frac{\sin(t(\omega - \lambda N)/2)}{\omega - \lambda N} \right)^2. \quad (12)$$

In the limit when the noise is sufficiently slow, i.e. $\omega_c \ll E_g$, then $\overline{\gamma^M(t)}$ is bounded for all times, $\overline{\gamma^M(t)} < \left(\frac{N^2 - 4M^2}{N^2} \right) \left(\frac{f\omega_c}{\lambda^2 N} \right) \ll 1$ and the atomic population within the ground state manifold is fully preserved i.e. $\gamma^M(t) \approx 0$ in Eq. (9).

Consequently, in the slow noise limit type (ii) processes are energetically forbidden and only type (i) processes are effective and therefore the noise acts just as a uniform random magnetic field: if at $t = 0$ $\hat{\rho} = \sum_{M,\tilde{M}} \rho_{M,\tilde{M}}(0) |\frac{N}{2}, \tilde{M}\rangle \langle \frac{N}{2}, M|$, then after time t each component $\rho_{M,\tilde{M}}$ acquires an additional random phase $e^{i(\theta_M(t) - \theta_{\tilde{M}}(t))}$ and on average

$$\overline{\rho_{M,\tilde{M}}(t)} = \rho_{M,\tilde{M}}(0) e^{i\chi t(M^2 - \tilde{M}^2)} e^{-\Gamma(t) \frac{(M - \tilde{M})^2}{N}}. \quad (13)$$

The factor of \sqrt{N} in the denominator of θ_M is fundamental for the reduction of the effect of decoherence within the MPM. For example, it makes $\hat{J}_{x,y}^{(0)}$ to decay N times slower than in the unprotected system: i.e. $\langle \hat{J}_{x,y}^{(0)}(t) \rangle = e^{-\Gamma(t)/N} \langle \hat{J}_{x,y}^{(0)}(t) \rangle |_{\Gamma=0}$.

Assuming all atoms are initially polarized in the x direction, $\rho_{M,\tilde{M}}(0) = 2^{-N} \sqrt{\binom{N}{M+N/2} \binom{N}{\tilde{M}+N/2}}$, using Eqs. (13), the approximation $\binom{N}{M+N/2} \approx \left(\frac{2}{\pi N}\right)^{1/4} e^{-\frac{M^2}{N}}$, valid in the large N limit and replacing the sums over M and \tilde{M} by integrals the fidelity at a given time t can be shown to be given by:

$$\mathcal{F}(t_0) = \frac{1}{\sqrt{1 + \Gamma(t_0)}}. \quad (14)$$

The insensitivity of $\mathcal{F}(t)$ on N , and the N times slower decay rate of $\langle \hat{J}_x^{(0)} \rangle$ demonstrate the usefulness of MPM to generate a large number of entangled particles.

C. Protection against arbitrary noise

We now discuss the protection against spin flips, which can be modeled by terms proportional to σ_i^x, σ_i^y in the noise Hamiltonian, Eq.(4). First of all note that as the \uparrow and \downarrow states have a finite energy splitting ω_0 , low frequency noise associated with such terms will be suppressed due it. However, most of the spin flips are generally induced by imperfections in the laser fields and therefore are at frequencies close to ω_0 , i.e. they correspond to low frequency noise in the rotating frame of the laser. In the case involving GHZ state generation, the finite energy cost imposed by χJ_z^2 between levels with different $|M|$ value tends to inhibit these processes as illustrated in Fig. 2. If in addition \hat{H}_{prot} is present this natural protection can be dramatically enhanced due to the fact that the energy gap suppresses local fluctuations in all directions. More precisely, noise modeled as $\sum_i h^\alpha \hat{\sigma}_i^\alpha$, when projected into the MPM reduces to $\frac{1}{\sqrt{N}} g^\alpha(0) \hat{J}_\alpha^0$ with $\alpha = x, y, z$.

In Fig. 4 we quantify the protection provided by the MPM by comparing the GHZ preparation fidelity with and without protection. In the Figure we also illustrate that within the MPM the effect of decoherence on the dynamics strongly depends on the direction of the noise. \hat{J}_x^0 and \hat{J}_y^0 noise degrade stronger the fidelity than \hat{J}_z^0 noise because while the latter commutes with \hat{H}_z and just imprints a global rotation of the Bloch vector, the former do not commute with it. Additionally we show that for noise with long correlation time the use of spin echo techniques can help to further reduce the effect of decoherence.

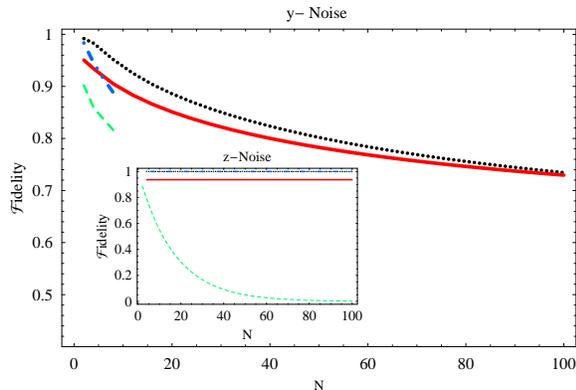


FIG. 4: Comparison of the fidelity to create GHZ state as a function of N for systems: without MPM (green dashed line), without MPM but with spin echo (dot-dashed blue line), with MPM (red solid line) and with both MPM and echo (dotted black line). In the main panel the noise was considered to be σ_y noise, in the inset we show the effect of σ_z noise for comparison purposes. For simplicity we assumed infinite correlation time: $f(\omega) = \Upsilon\delta(\omega/\chi)$, $\Upsilon = 0.1\chi$. The echo technique consisted of a perfect sudden π pulse around the x direction at $\chi t = \pi/4$. Because the y components of the noise do not commute with $\hat{J}_z^{2(0)}$ the dynamics was solved numerically. For the unprotected system all the 2^N states had to be considered and we had to limit the particle number by $N = 10$. On the other hand the restriction of the dynamics to the MPM in the protected scheme allowed us to extend the calculation to larger N values. For the σ_y noise the fidelity does not become N independent, nevertheless it is significantly enhanced with respect to the unprotected system. This plot shows that the combination of MPM with spin echo provides the best protection.

IV. APPLICATIONS TO PRECISION MEASUREMENTS USING TRAPPED IONS

Recent experiments [1, 2] have generated GHZ states made of up to six beryllium ions and used them to perform precision measurements of ω_0 . For the ideal GHZ state preparation, the spectroscopy should lead to Heisenberg-limited resolution, $|\delta\omega_0| \propto N^{-1}$ [11]. However, in practice, even for six ions, the phase accuracy was significantly degraded by decoherence.

The spectroscopy [1, 2] was realized by first creating the desired GHZ state by applying to the initial polarized state, $|J = N/2, N/2\rangle_z$ the unitary gate operation $U_N = e^{i\pi/2\hat{J}_y^{(0)}} e^{i\pi/2\hat{J}_z^{(0)2}} e^{-i\pi/2\hat{J}_y^{(0)}}$. Then the GHZ state was let to freely precess in the z direction for time t so each atom accumulated a phase difference $\phi = (\omega - \omega_0)t$ (in a reference frame rotating with the frequency ω , the frequency of the applied field). The phase difference was then decoded by measuring the collapse probability into the states $|J = N/2, N/2\rangle_z$ or $|J = N/2, -N/2\rangle_z$ after applying the unitary transformation $\prod_i \sigma_i^x$.

The generalized Ramsey sequence done can be quantitatively described as a measure of the expectation value of the

following operator, \hat{O} :

$$\langle \hat{O} \rangle = \langle \psi(t_0) | \prod_i [\cos(\phi)\hat{\sigma}_i^z - \sin(\phi)\hat{\sigma}_i^y] | \psi(t_0) \rangle \quad (15)$$

with $|\psi(t_0)\rangle = e^{-i\frac{\pi}{2}\hat{J}_z^{(0)2}} |N/2, N/2\rangle_x$.

The phase sensitivity $\langle \Delta^2 \hat{O} \rangle$ achievable by repeating the above scheme during total time T is related to the signal variance $\langle \Delta^2 \hat{O} \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$ and given by:

$$|\delta\omega_0| = \sqrt{\frac{1}{iT} \frac{\langle \Delta^2 \hat{O} \rangle}{(\delta\langle \hat{O} \rangle / \delta\phi)^2}} \quad [12].$$

Because $\hat{O}^2 = 1$, we just have to calculate $\langle \hat{O} \rangle = \text{Tr}[\hat{\rho}(t_0)\hat{O}]$ to evaluate $\delta\omega_0$.

Experimentally, magnetic field noise is one of the main sources of phase decoherence. Assuming that such dephasing mainly takes place during the GHZ generation, as during the Ramsey interrogation time the atoms are essentially freely evolving, using Eq.(5) one can show that for the unprotected system

$$\begin{aligned} \langle \hat{O} \rangle &= \langle \psi_x^{GHZ} | \prod_i [\cos(\phi)\hat{\sigma}_i^z - e^{-\Gamma(t_0)} \sin(\phi)\hat{\sigma}_i^y] | \psi_x^{GHZ} \rangle \\ &= \frac{1}{2} \left[\frac{e^{i\phi}(1 - e^{-\Gamma(t_0)}) + e^{-i\phi}(1 + e^{-\Gamma(t_0)})}{2} \right]^N + h.c. \end{aligned} \quad (16)$$

Consequently the maximal phase resolution, achieved at $\phi_{opt} = n\pi$ (for integer n), can be shown to be given by

$$|\delta\omega_0|_{opt} = |\delta\omega_0|_{sh}/G, \quad (17)$$

with $G = \sqrt{((N-1)e^{-2\Gamma(t_0)} + 1)}$ and $|\delta\omega_0|_{sh} = \frac{1}{\sqrt{tTN}}$ the shot noise resolution. The factor G explains the strong limitations introduced by decoherence. If the time required to generate the GHZ state is such that $G \sim 1$ (i.e when $\Gamma(t_0) > \ln[\sqrt{N}]$), the phase accuracy is reduced to the classical shot noise resolution.

However, if instead $\hat{H}_{prot} + \hat{H}_z$ is used for the GHZ generation, G is replaced by $\sqrt{((N-1)e^{-2\Gamma(t_0)/N} + 1)}$. Due to the N times slower decay rate of the atomic coherences, the same preparation time that leads to shot noise resolution without protection, can lead to a sensitivity at the level of the fundamental Heisenberg limit with protection.

V. IMPLEMENTATION OF THE GAP PROTECTED HAMILTONIAN IN TRAPPED IONS

Here we now proceed to review and complement the implementation of the protected Hamiltonian, $\chi(\hat{J}^{(0)2} - \hat{J}_z^{(0)2})$, proposed in Ref.[14]. Consider a linear trap with a string of ions with two relevant internal levels. The ions are assumed to be cooled such that only the in-phase collective center of mass oscillation of all ions is excited. The corresponding oscillation frequency is denoted by ν . The two internal levels of the ions are coupled by a laser field with a slowly varying Rabi frequencies Ω and with frequency $\omega_1 = \omega_o - \delta$, being δ the detuning from resonance. Assuming that the field couple

all ions in the same way, we can describe the system by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{in}$, where $H_0 = \nu\hat{a}^\dagger\hat{a} + \omega_0\hat{j}_z^{(0)}$, \hat{a} being the annihilation operator of the quantized oscillation mode. The interaction Hamiltonian \hat{H}_{in} is given by

$$\hat{H}_{in} = \Omega\hat{j}_+^{(0)}e^{i\delta t}e^{i\eta(\hat{a}^\dagger e^{i\nu t} + \hat{a}e^{-i\nu t})} + h.c. \quad (18)$$

where η is the Lamb-Dicke parameter. The detuning δ is assumed to be large compared to the linewidth of the resonance but sufficiently different from the frequency of the trap oscillation. As a consequence, the dominant processes are two-photon transitions leading to a simultaneous excitation of pairs of ions.

We first assume that the ion trap is in the Lamb-Dicke limit, i.e., that the ions are cooled sufficiently enough, such that for all relevant excitation numbers n of the trap oscillation $(n+1)\eta^2 \ll 1$ holds. In this limit one can expand the exponent in Eq. (18) to first order in η . Confining the interest to time averaged dynamics over a period much longer than any of the oscillations present in \hat{H}_{in} , then the oscillatory terms may be neglected and we are left with a more simple effective Hamiltonian Ref.[15] :

$$H_{eff} = \chi(\hat{j}^{(0)2} - \hat{j}_z^{(0)2}) + \frac{2\Omega^2}{\delta}\hat{j}_z^{(0)} + \lambda(2n+1)\hat{j}_z^{(0)} \quad (19)$$

where $\chi = \frac{2\nu\eta^2\Omega^2}{\delta^2 - \nu^2}$, $\lambda = \frac{\chi\delta}{\nu}$ and n the number of phonons in the vibrational mode. The first term in H_{eff} is the desired protected Hamiltonian. The second term acts as an effective magnetic field which can be canceled by adding an external magnetic field or by echo techniques. The third term comes from the ac Stark shift of the atomic levels due to the laser fields. In contrast to the standard scheme used to create $\hat{j}_z^{(0)2}$, where the n dependence exactly cancels, here it does not and if not corrected can certainly degrade the fidelity. The degradation can be shown to be given by:

$$\mathcal{F}(t_o) = \sum_n P_n \exp\left[-\frac{N^2\lambda^2(2n+1)^2t_o^2}{8}\right] \quad (20)$$

where P_n is the initial population of the state with n phonons.

In order to prevent this effect one has to cool the ions to the ground state, $P_n = \delta_{n0}$, which might be feasible with the state of the art technology or alternatively one can use spin echo techniques. For example if at time $t_o/2$ the sign of the laser detuning δ is changed, then the different components will rotate in the opposite direction and at t_o the net effect due to the extra second and third terms in \hat{H}_{eff} will be canceled out.

So far we have used the Lamb-Dicke and the rotating-wave approximation. Now we perform a more detailed analysis of the validity of these approximations and estimate the effect of deviations from the ideal situations in an actual experiment. To do that we follow Ref. [8] and change to the interaction picture of \hat{H}_{eff} , assuming that the undesired second and third terms can be canceled by the techniques described above, and treat the small non-ideal deviations by perturbation theory.

• Direct coupling

Going from Eq.(18) to Eq.(19) the off-resonant term $H_d = \Omega\hat{j}_+^{(0)}e^{it\delta} + h.c.$ was neglected. This term corresponds to direct single atom spin flips without any vibrational excitation.

Changing to the interaction picture of H_{eff} and using the fact that H_d oscillates at a much higher frequency than $\hat{U}(t) = e^{iH_{eff}t}$ so that the latter can be treated as constant in the integrals used in the Dyson series, one can show that

$$\mathcal{F}(t_o) = 1 - \frac{\Omega^2}{\delta^2} (N^2 \sin^2(\delta t_o) + 4N \sin^4(\delta t_o/2) + \dots) \quad (21)$$

The degradation of fidelity is a factor of N larger than the degradation caused by direct couplings in the standard realization of $J_z^{(0)2}$ where $\mathcal{F}(t_o) = 1 - \frac{N\Omega^2}{\delta^2} \sin^2(\delta t_o)$. Therefore it is important for the implementation of the protected Hamiltonian to use weak laser power or to control the system parameter such that $\delta t_o = 2K\pi$ with K an integer.

• Lamb-Dicke approximation

In Ref.[8] it has been shown that relaxing the Lamb-Dicke approximation and including higher order terms results in an effective χ_n which depends on the vibrational number of phonons in the collective mode: $\chi_n = \chi[1 - \eta^2(2n+1) + \eta^4(5/4n^2 + 5/4n + 1/2)]$. As this effect is global, the gap does not protect against it and it leads to a degradation of the fidelity given by

$$\begin{aligned} \mathcal{F}(t_o) &= \sum_n P_n \left(1 + \frac{N(N-1)(\pi/2 - \chi_n t_o)^2}{4}\right)^{-1/2} \\ &\sim 1 - \frac{\pi^2 N(N-1)\eta^4}{32} \sum_n P_n (2n+1)^2 \end{aligned} \quad (22)$$

• Other vibrational modes

With N ions in the trap, assuming that the transversal potential is strong enough to freeze the transversal degree of freedom, only the N longitudinal vibrational modes are relevant. So far we have assumed that only the collective center of mass motion is excited and neglected other modes. If we include the effect of other modes into account, the fidelity is decreased. The main sources of decoherence are :a) off resonant direct couplings to other modes and b) reduction of the coupling to the center of mass mode, χ , due to the vibration of the other modes. However all these effects are local and the gap energetically suppresses them.

• Spontaneous emission

Additionally a more fundamental source of decoherence arises from spontaneous emission effects. In typical ion trap experiments the $|\uparrow\rangle$ and $|\downarrow\rangle$ levels are coupled through Raman transitions to a third excited level $|e\rangle$ (see Fig.5). Assuming two photon resonance conditions, that is $\omega_{e1} - \omega_{e2} = \nu_1 - \nu_2 = \omega_0$ and $\Delta = \omega_{e1} - \nu_1 = \omega_{e2} - \nu_2$ where Δ is

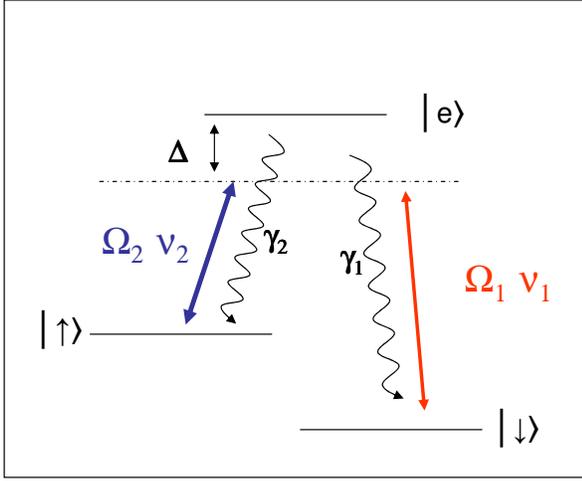


FIG. 5: Three level atom with atomic decay

the detuning of the fields from the one photon resonance and $\omega_{e1,2}$ and $\Omega_{1,2}$ are laser frequencies and Rabi frequencies respectively, the Hamiltonian of the system in the appropriate rotating frame can be written as

$$\hat{H}_s = -\Delta \sum_i \hat{\sigma}_{ee}^i + \hat{H}_{Is} \quad (23)$$

$$\hat{H}_{Is} = \Omega_1 \sum_i (\hat{\sigma}_{\downarrow e}^i + \hat{\sigma}_{e\downarrow}^i) + \Omega_2 \sum_i (\hat{\sigma}_{\uparrow e}^i + \hat{\sigma}_{e\uparrow}^i) \quad (24)$$

where $\hat{\sigma}_{ee}^i = |e\rangle_{ii}\langle e|$, $\hat{\sigma}_{\downarrow e}^i = |\downarrow\rangle_{ii}\langle e|$ and $\hat{\sigma}_{\uparrow e}^i = |\uparrow\rangle_{ii}\langle e|$.

The decoherence processes due to spontaneous emission can be described by means of Heisenberg-Langevin equations[17] given by:

$$\dot{\hat{\sigma}}_{\uparrow\downarrow}^j = (i\Omega_1 + \hat{f}_{\downarrow e}^{\dagger j})\hat{\sigma}_{\uparrow e}^j - (i\Omega_2 - \hat{f}_{\uparrow e}^j)\hat{\sigma}_{e\downarrow}^j \quad (25)$$

$$\begin{aligned} \dot{\hat{\sigma}}_{\uparrow e}^j &= -(i\Delta + \Gamma_e/2)\hat{\sigma}_{\uparrow e}^j - (i\Omega_2 - \hat{f}_{\uparrow e}^j)(\hat{\sigma}_{ee}^j - \hat{\sigma}_{\uparrow\uparrow}^j) \\ &\quad + (i\Omega_1 - \hat{f}_{\downarrow e}^j)\hat{\sigma}_{\uparrow\downarrow}^j \end{aligned} \quad (27)$$

$$\begin{aligned} \dot{\hat{\sigma}}_{e\downarrow}^j &= (i\Delta - \Gamma_e/2)\hat{\sigma}_{e\downarrow}^j + (i\Omega_1 + \hat{f}_{\downarrow e}^{\dagger j})(\hat{\sigma}_{ee}^j - \hat{\sigma}_{\downarrow\downarrow}^j) \\ &\quad - (i\Omega_2 + \hat{f}_{\uparrow e}^{\dagger j})\hat{\sigma}_{\uparrow\downarrow}^j \end{aligned} \quad (29)$$

where $\Gamma_e = \gamma_1 + \gamma_2$ with γ_1 and γ_2 are decay rates from $|e\rangle$ to $|\downarrow\rangle$ and $|\uparrow\rangle$ respectively and the noise operators \hat{f} have zero mean and are δ correlated[17]: $\langle \hat{f}_{\downarrow e}^j(t) \hat{f}_{\downarrow e}^{k\dagger}(t') \rangle = \gamma_1 \delta(t-t') \delta_{j,k}$ and $\langle \hat{f}_{\uparrow e}^j(t) \hat{f}_{\uparrow e}^{k\dagger}(t') \rangle = \gamma_2 \delta(t-t') \delta_{j,k}$.

In the large photon detuning limit $\Delta \gg \Omega_1 \Omega_2, \gamma_1, \gamma_2$, one can adiabatically eliminate the operators $\hat{\sigma}_{e\downarrow}^j$ and $\hat{\sigma}_{e\uparrow}^j$ and their hermite conjugates and then use the projected equations of motion to solve for $\hat{\sigma}_{\uparrow\downarrow}^j$.

$$\dot{\hat{\sigma}}_{\uparrow\downarrow}^j = FH + i[\hat{H}_{\text{noise}}, \hat{\sigma}_{\uparrow\downarrow}^j] \quad (30)$$

$$\hat{H}_{\text{noise}}(t) = \frac{1}{2} \sum_j [h_{jz}^s(t) \hat{\sigma}_j^z + h_{jx}^s(t) \hat{\sigma}_j^x + h_{jy}^s(t) \hat{\sigma}_j^y] \quad (31)$$

where FH accounts for the Hamiltonian part of the dynamics and with $h_{jz}^s(t) = \frac{\Omega_1}{i\Delta}(\hat{f}_{\downarrow e}^{j\dagger} - \hat{f}_{\downarrow e}^j) - \frac{\Omega_2}{i\Delta}(\hat{f}_{\uparrow e}^{j\dagger} - \hat{f}_{\uparrow e}^j)$, $h_{jx}^s(t) = -\frac{\Omega_2}{i\Delta}(\hat{f}_{\downarrow e}^{j\dagger} - \hat{f}_{\downarrow e}^j) - \frac{\Omega_1}{i\Delta}(\hat{f}_{\uparrow e}^{j\dagger} - \hat{f}_{\uparrow e}^j)$ and $h_{jy}^s(t) = -\frac{\Omega_2}{\Delta}(\hat{f}_{\downarrow e}^{j\dagger} + \hat{f}_{\downarrow e}^j) + \frac{\Omega_1}{\Delta}(\hat{f}_{\uparrow e}^{j\dagger} + \hat{f}_{\uparrow e}^j)$.

From the previous expressions one can estimate the degradation of the fidelity due to dephasing (similar degradation of the fidelity is caused by x or y type of noise). In the adiabatic limit, i.e. $\Delta \gg \Omega_1 \Omega_2, \gamma_1, \gamma_2$, h_{jz}^s are independent stochastic white noise processes with zero mean and autocorrelation function $\overline{h_{iz}^s(t) h_{jz}^s(\tau)} = \gamma_s \delta_{ij} \delta(t - \tau)$ with $\gamma_{sp} = \frac{(\gamma_1 \Omega_1^2 + \gamma_2 \Omega_2^2)}{\Delta^2}$ and consequently the gap can not protect against this broad band noise. From Eq.(6) they will cause a degradation of the GHZ fidelity of

$$\mathcal{F} = 1 - \gamma_{sp} N t_o \quad (32)$$

In order to reduce the strong degradations due to this type of high-frequency decoherence processes one can increase the Raman-detuning [16] at the expense of slower evolution which in turn will make the system more susceptible to other kind of local noise (eg. magnetic field inhomogeneities). On the other hand the latter can be suppressed by the MPM.

From this analysis we conclude that overhead of implementing $\hat{J}^{(0)2} - \hat{J}_z^{(0)2}$ instead of $\hat{J}_z^{(0)2}$ is mainly the additional echo technique required to remove the n dependence of H_{eff} . Besides that, on average the same type of non-ideal disturbances are found in both Hamiltonians with the advantage of $\hat{J}^{(0)2} - \hat{J}_z^{(0)2}$ that the gap protects the system against those of them which are local in character.

VI. IMPLEMENTATION IN OPTICAL LATTICES

A. Engineering long-range interactions

Up to now we have explored only the generation of an MPM via isotropic long-range interactions. In practice, however, it is desirable to have a similar kind of protection generated by systems with short range interactions such as those provided by cold atoms in optical lattices. These systems offer the possibility to dynamically change the Hamiltonian parameters at a level unavailable in more traditional condensed matter systems. We now show how an MPM can be created in lattice systems and can be used to robustly generate N-particle GHZ states.

We consider ultracold bosonic atoms with two relevant internal states confined in an optical lattice. We will assume that the lattice is loaded with one atom per site, and again identify the two possible states of each site, with the effective

spin index $\sigma = \uparrow, \downarrow$ respectively. For deep periodic potential and low temperatures, the atoms are confined to the lowest Bloch band and the low energy Hamiltonian is given by [18]

$$\hat{H}_{BH} = -\tau \sum_{\langle i,j \rangle \sigma} \hat{a}_{\sigma,i}^\dagger \hat{a}_{\sigma,j} + \frac{1}{2} \sum_{j\sigma} U_{\sigma\sigma} \hat{n}_{\sigma,j} (\hat{n}_{\sigma,j} - 1) + U_{\uparrow\downarrow} \hat{n}_{\sigma,j} \hat{n}_{\sigma,j} \quad (33)$$

Here $\hat{a}_{\sigma,j}$ are bosonic annihilation operators of a particle at site j and state σ , $\hat{n}_{\sigma,j} = \hat{a}_{\sigma,j}^\dagger \hat{a}_{\sigma,j}$, and the sum $\langle i,j \rangle$ is over nearest neighbors. In Eq.(33) the parameter τ is the tunneling energy between adjacent sites (which we assume spin independent i.e. spin independent lattices) and $U_{\sigma\sigma'}$ are the different on-site interaction energies which depend on the scattering length between the different species. Both $U_{\sigma\sigma'}$ and τ are functions of the lattice depth. We are interested in a unit filled lattice in the regime $\tau \ll U_{\sigma\sigma'}$ where the system is deep in the Mott insulating phase [19, 20]. In this limit, to zero order in τ the ground state is multi-degenerate and corresponds to all possible spin configuration with one atom per site. A finite τ breaks the spin degeneracy. By including virtual particle-hole excitations one can derive an effective Hamiltonian that describe the spin dynamics within the one atom per site subspace[21]:

$$\hat{H}_{lat} = \hat{H}_H + \hat{H}_I = -\bar{\lambda} \sum_{\langle i,j \rangle, \alpha} \hat{\sigma}_i^\alpha \hat{\sigma}_j^\alpha - \bar{\chi} \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (34)$$

Here the coefficients are $\bar{\lambda} = \tau^2/U_{\uparrow\downarrow}$ and $\bar{\chi} = \tau^2(U_{\uparrow\uparrow}^{-1} + U_{\downarrow\downarrow}^{-1} - 2U_{\uparrow\downarrow}^{-1})$. For simplicity we will now restrict the analysis to one dimensional systems and assume periodic boundary conditions.

\hat{H}_H is spherically symmetric and in terms of collective spin operators it can be written as

$$\hat{H}_H = -\frac{4\bar{\lambda}}{N} \hat{J}_z^{(0)2} - \frac{4\bar{\lambda}}{N} \sum_{k=1 \dots N-1, \alpha} \hat{J}_\alpha^{(k)} \hat{J}_\alpha^{(-k)} \cos\left(\frac{2\pi k}{N}\right) \quad (35)$$

All the $N+1$ fully symmetric states with $J = N/2$ are degenerate and span the ground state of \hat{H}_H . \hat{H}_I is not spherically symmetric but we can also write it in terms of collective operators as

$$\hat{H}_I = -\frac{4\bar{\chi}}{N} \hat{J}_z^{(0)2} - \frac{4\bar{\chi}}{N} \sum_{k=1 \dots N-1} \hat{J}_z^{(k)} \hat{J}_z^{(-k)} \cos\left(\frac{2\pi k}{N}\right). \quad (36)$$

If the condition $\bar{\chi} \ll \bar{\lambda}$ is satisfied, which can be engineered in this atomic systems by means of a Feshbach resonance, the effect of the Ising term can be studied by means of perturbation theory. Assuming that at $t = 0$ the initial state is prepared within the $J = N/2$ manifold, a perturbative analysis predicts that for times t such that $\bar{\chi}t < \bar{\lambda}/\bar{\chi}$, \hat{H}_H confines the dynamics to the ground state manifold and transitions outside it can be neglected. As a consequence, only the projection of \hat{H}_I on it, which corresponds to $\mathcal{P}\hat{H}_I = \chi_e \hat{J}_z^{(0)2} - \frac{\bar{\lambda}N}{N-1}$ with $\chi_e \equiv \frac{4\bar{\chi}}{N-1}$, is effective and H_I acts as a long range

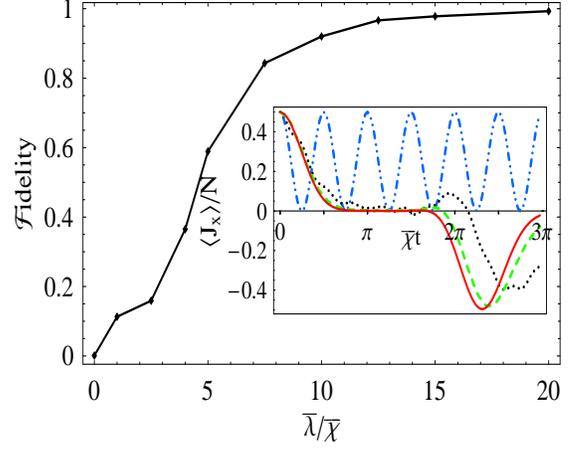


FIG. 6: (color online) Fidelity to generate a GHZ state vs $\bar{\lambda}/\bar{\chi}$. In the inset we show $\langle \hat{J}_x^{(0)} \rangle$. The blue dot-dashed, dotted black, dashed green and solid red curves correspond to $\bar{\lambda} = 0, 5, 10, 20$ respectively. The plots are obtained by numerical evolution of Eq. (34) for $N = 10$.

Hamiltonian. Here we used the relation $\mathcal{P}_{k \neq 0} [\hat{J}_z^{(k)} \hat{J}_z^{(-k)}] = -\frac{\hat{J}_z^{(0)2}}{N-1} + \frac{N^2}{4(N-1)}$, with \mathcal{P} the projection into the $J = N/2$ subspace. The non zero projection of the latter term comes from the fact that the operators $\hat{J}_z^{(k)} \hat{J}_z^{(-k)}$ and $\hat{J}_z^{(0)2}$ are not independent as they satisfy the constrain $\sum_{k=0}^{N-1} \hat{J}_z^{(k)} \hat{J}_z^{(-k)} = N^2/4$.

In Fig. 6 we contrast the dynamical evolution of a system in the presence and absence of \hat{H}_H assuming at time $t = 0$ all the spins are polarized in the x direction. If only the Ising term is present, $\bar{\lambda} = 0$, it induces local phase fluctuations that leads to fast oscillations in $\langle \hat{J}_x^{(0)} \rangle = N/2 \cos^2[2\bar{\chi}t]$. On the other hand, as the ratio $\bar{\lambda}/\bar{\chi}$ increases, the isotropic interaction inhibits the fast oscillatory dynamics and instead $\langle \hat{J}_x^{(0)} \rangle$ exhibits slow collapses and revivals. For $\bar{\lambda} \gg \bar{\chi}$ the dynamics exactly resembles the one induced by \hat{H}_z and at $\chi_e t = \pi/2$ the initial coherent state is squeezed into a GHZ state.

B. MPM in lattice systems

\hat{H}_H also provides protection against phase decoherence. However, \hat{H}_H is not as effective as \hat{H}_{prot} because the energy gap between the MPM and the excited states of \hat{H}_H vanishes in the thermodynamic limit as $E_g \rightarrow \bar{\lambda}/N^2$. This is a drawback of the short range Hamiltonian for the purpose of fully protecting the ground states from long wave length excitations. Note however that one dimensional systems are the worst scenario as for higher dimensions the gap vanishes as $N^{-2/d}$ with d the dimensionality of the system. Nevertheless, the many body interactions can still eliminate short-wavelength excitations since in the large N limit they remain separated by a finite energy gap, $8\bar{\lambda}$.

We quantify the effectiveness of the MPM to protect the system against \hat{H}_{env} by using time dependent perturbation

theory. For this analysis we restrict to the limit $\bar{\lambda} \gg \bar{\chi}$ where the Ising term can be treated as an effective $\chi e^{\hat{J}_z^{(0)2}} - \frac{\bar{\lambda}N}{N-1}$ Hamiltonian. In this limit a convenient basis to study the quantum dynamics is the collective spin basis. Assuming that at $t = 0$ the system lies within the $J = N/2$ manifold, the evolution of the matrix elements $\rho_{M\bar{M}} \equiv {}_z\langle N/2, \bar{M} | \hat{\rho} | N/2, M \rangle_z$ can be written as

$$\rho_{M,\bar{M}}(t) = \rho_{M,\bar{M}}(0) e^{it\chi_e(M^2 - \bar{M}^2)} e^{i(\theta_M - \theta_{\bar{M}})} e^{-\frac{1}{2}(\gamma_{lat}^M - \gamma_{lat}^{\bar{M}})} \quad (37)$$

where the random phase, given by Eq.(10), characterizes the dynamics induced by the noise within the MPM and $\gamma_{lat}^M(t) = \sum_{J \neq N/2, \beta} \left| \int_0^t d\tau \mathcal{M}_{J,\beta}^M e^{i\tau\omega_{J,\beta}^{lat}} \right|^2$, takes into account the depletion of the $J = N/2$ levels due to transition matrix elements with states outside the symmetric manifold: $\mathcal{M}_{J,\beta}^M = {}_z\langle \frac{N}{2}, M | \hat{H}_{env} | J, M, \beta \rangle_z$. $\omega_{J,\beta}^{lat}$ are the respective energy splittings. Up to this point the expressions are structurally identical to the ones obtained for long range interactions. The difference appears in the evaluation of γ_{lat} . In contrast to \hat{H}_{prot} , not only the excitation frequencies $\omega_{J,\beta}^{lat}$ are not degenerated but also they become smaller as N is increased. As a consequence Eq. (12) is replaced by the following equation for the lattice system

$$\overline{\gamma_{lat}^M(t)} \approx \frac{N^2 - 4M^2}{N(N-1)} f \sum_{k=1}^{N-1} \int_0^{\omega_c} d\omega \left(\frac{\sin(t(\omega - \Delta E^k)/2)}{\omega - \Delta E^k} \right)^2 \quad (38)$$

with E^k the excitation energies of the states that belong to the $J = N/2 - 1$ manifold given by $\Delta E^k = 8\bar{\lambda} \sin^2(\pi k/N)$, with $k = 1, \dots, N-1$ [22]. From Eq.(38) we can estimate the degradation of the fidelity due to phase decoherence as:

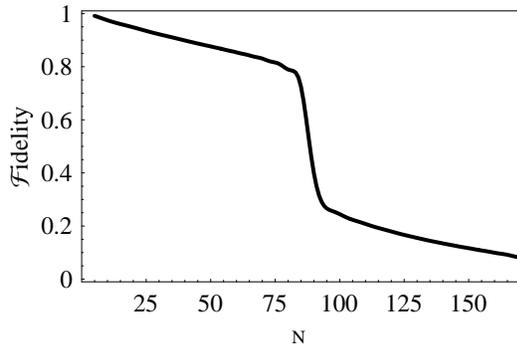


FIG. 7: In the presence of phase decoherence the fidelity of the GHZ state preparation in the lattice is degraded as N grows because in the lattice the gap decreases and the generation time increases with N . In this plot we assumed the limit $\bar{\chi} \ll \bar{\lambda}$ where Eq. (39) holds and used a system with $\omega_c = \bar{\chi}$, $\bar{\lambda} = 100\bar{\chi}$ and $\Gamma = 0.01\bar{\chi}$. At $N = 90$, $\Delta E_g = \omega_c$ and it explains the drop of \mathcal{F} for $N > 90$.

$$\mathcal{F}(t_o) \gtrsim e^{\overline{\gamma_{lat}^0(t_o)}} \frac{1}{\sqrt{1 + \Gamma(t_o)}}. \quad (39)$$

In Fig.7 we plot \mathcal{F} calculated from Eq.(39) as a function of N . In the lattice the fidelity is degraded as N grows because the gap decreases and the generation time increases with N . Moreover, an abrupt drop of the fidelity occurs at the value of N at which $E_g = \omega_c$.

VII. NOISE AND DECOHERENCE IN LATTICE SYSTEMS

In the previous section we used the effective Hamiltonian given by Eq. (34) to study the GHZ generation in lattice systems. Here we perform a more detailed analysis of its validity and estimate the effect of deviations from the ideal situations in an actual experiment. For this analysis we restrict to the limit $\bar{\lambda} \gg \bar{\chi}$ where the Ising term can be treated as an effective $\chi e^{\hat{J}_z^{(0)2}} - \frac{\bar{\lambda}N}{N-1}$ Hamiltonian.

A. Particle-hole excitations

Deriving Eq. (34) from the Bose-Hubbard Hamiltonian we only included virtual-particle hole excitation. However, during the time evolution real transitions from singly to doubly occupied states can take place and they degrade the fidelity.

To account for these effects, we write the manybody wave function as $|\Psi(t)\rangle = \sum_n C_n |\psi_n\rangle + \sum_m B_m |\phi_m\rangle$, where $|\psi_n\rangle$ span the Hilbert space with one atom per site and $|\phi_m\rangle$ span the subspace with one particle and one hole adjacent to each other and $N-2$ singly occupied sites. The latter are the states that directly coupled to $|\psi_n\rangle$ through tunneling. Solving the time dependent Schrödinger equation from the Bose Hubbard Hamiltonia, using the assumption that $U_{\sigma,\sigma'} \approx U$ and that at time $t = 0$ no doubly occupied states are populated one obtains

$$i\dot{C}_n = \sum_k \langle \psi_n | \hat{H}_{latt} | \psi_k \rangle (1 - e^{-iUt}) C_k \quad (40)$$

Here we also assumed that $\{C_n\}$ change at a rate much smaller than $U_{\sigma,\sigma'}$ and treated them as constants during the time integration. Eq.(40) yields the following lost of fidelity due to real particle hole excitations:

$$\mathcal{F}(t_o) \approx 1 - \frac{4\bar{\chi}}{U} \sin^2(Ut_o/2). \quad (41)$$

remembering that $\chi_e t_o = \pi/2$. As long as $\bar{\chi}/U \ll 1$, we conclude that particle-hole excitations do not significantly affect the GHZ generation.

B. Magnetic confinement

In Eq. (33) we assumed a translationally invariant system. However in most of the experiments an additional quadratic magnetic confinement is used to collect the atoms. Actually is due to this quadratic potential that a unit filled Mott insulator

has been experimentally realized. In its absence it would be difficult to create a unit filled Mott insulator as in an homogeneous system it only takes place when the number of atoms is exactly equal to the number of lattice sites. A drawback of the magnetic confinement is that it generates always superfluid regions at the edge of the cloud, so only a fraction of the total trapped atoms located at the trap center has to be selected as the quantum register. Assuming we work on this unit filled Mott Insulator subspace, here we quantify the effect of the magnetic potential in the GHZ generation in the $\bar{\lambda}/\bar{\chi} \gg 1$ limit.

The magnetic confinement is accounted for by adding a term $W \sum_{j,\sigma} j^2 \hat{n}_{\sigma,j}$ in the Bose Hubbard Hamiltonian. $W = 1/2m\omega_T^2 a_L^2$ with m the atom mass, ω_T the frequency of the external trapping potential and a_L the lattice spacing. This term modifies the global coupling constants $\bar{\lambda}$ and $\bar{\chi}$ when the effective Hamiltonian is derived and make them site dependent, $\bar{\lambda} \rightarrow \bar{\lambda}_i^W \equiv \tau^2/\tilde{U}_{i,\uparrow\downarrow}$ and $\bar{\chi} \rightarrow \bar{\chi}_i \equiv \tau^2(\tilde{U}_{i,\uparrow\uparrow}^{-1} + \tilde{U}_{i,\downarrow\downarrow}^{-1} - 2\tilde{U}_{i,\uparrow\downarrow}^{-1})$. Here $\tilde{U}_{i,\sigma\sigma'} = U_{\sigma\sigma'}/(U_{\sigma\sigma'}^2 - W^2(2i+1)^2)$. Assuming that the gradient of the external potential is weak compared to the on site interaction energy, as is in general the case for current experiments, the effective Hamiltonian in the presence of the magnetic trap becomes

$$\hat{H}_{lat}^W = \hat{H}_{lat} + \hat{H}_1^T \quad (42)$$

$$\hat{H}_1^T = - \sum_{\langle i,j \rangle} T_i \vec{\sigma}_i \cdot \vec{\sigma}_j \quad (43)$$

$$T_i = - \frac{\tau^2 W^2 (2i+1)^2}{U^3} \quad (44)$$

The corrections on the fidelity of the GHZ state introduced by \hat{H}_1^T can be estimated by calculating the effective projection of it on the MPM: $\mathcal{P}\hat{H}_1^T$. As the latter is just proportional to the identity matrix I , $\mathcal{P}\hat{H}_1^T = \sum_i T_i I$, its effect is just a global phase and it does not cause any main degradation of the fidelity. Similarly any other perturbation induced by local fluctuations in the magnetic field or the lasers used to generate the lattice become irrelevant thanks to the MPM.

From this analysis we conclude that except from spontaneous emission or heating mechanisms the MPM effectively protects lattice systems against common non-ideal situations encountered during their experimental realization. On the other hand lattice-based GHZ state generation faces the scalability problem due to the fact that the gap decreases and the generation time increases with increasing N .

VIII. CONCLUSIONS

We have in this paper evaluated the possibility for a robust preparation of multi-particle GHZ entangled states of trapped ions or cold atoms in optical lattice by generating a decoherence free multi-level manifold corresponding to the ground levels of properly designed Hamiltonians. The MPM is isolated from the rest of the Hilbert space by an energy gap which energetically suppresses any local decoherence processes. We have presented analytical estimates for the fidelity of the GHZ preparation.

In trapped ions we demonstrated that the fidelity can be significantly better than the one achievable without any gap protection and therefore that our scheme is in the position to improve the spectroscopy resolution in current Ramsey spectroscopy experiments.

We also showed that cold atoms in optical lattices interacting via short range interactions can be utilized to engineer long range interactions which in turn can be used for generating many-body entanglement. We calculated the effects of non-ideal conditions and concluded that the main restriction in these systems is the scalability as the MPM protection degrades with increasing N .

The scalability certainly limits the use of lattice systems for massive entanglement generation, however it is not a problem for recent quasi-one dimensional experiments [23] where an array of 1D tubes with an average of 18 atoms per tube has been realized. In such systems therefore it should be possible to create few-particle collective entangled states using our scheme and to perform proof-of-principle experiments demonstrating the improvement of spectroscopic sensitivity.

We emphasize that, even though we have limited the discussion to ensembles of spin $S = 1/2$ particles, the MPM ideas can be straightforwardly generalized to systems composed of higher spin atoms. Besides entanglement generation, the MPM might have also important applications for the implementation of good storage memories using for example nuclear spin ensembles in solid state [24] or photons[25].

IX. ACKNOWLEDGEMENTS

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