Luttinger liquids
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I. LUTTINGER LIQUIDS: A STRONG COUPLING APPROACH

We consider identical interacting particles in one dimension (either fermions or bosons). The Hamiltonian is given by

\[ H = \sum_n \frac{p_n^2}{2m} + V(x_1, \ldots, x_N), \]  

where \( V(x_1, \ldots, x_n) \) is a general interaction potential, which satisfies

\[ V(x_1, \ldots, x_N) = V(x_1 + a, \ldots, x_N + a) \]  

for an arbitrary translation \( a \). We imagine a potential which is minimal for a regular crystalline configuration, with \( x_n = nd \), where \( d \) is the average inter-particle spacing. We define the particle density \( \rho_0 = 1/d \).

The density operator is given by

\[ \rho = \sum_n \delta(x - x_n). \]  

In order to go to the continuum limit, we introduce a continuous “counting field” \( \tilde{\phi}(x) \) which satisfies:

\[ \tilde{\phi}(x_n) = 2\pi n \]  

A graphic description of this field is given in Fig. 1. It takes the value \( 2\pi n \) at the coordinate of the \( n \)th particle. For a perfect crystalline configuration, \( \phi(x_n) = 2\pi \rho_0 x_n \).

FIG. 1: From Giamarchi’s book (see ref. below). Different configurations of the counting field \( \tilde{\phi} \) (which Giamarchi denotes as \( \phi_1 \)). Left: crystalline configuration, right: general configuration.

In terms of \( \tilde{\phi}(x_n) \), one can write the density operator as

\[ \rho(x) = \nabla \tilde{\phi} \sum_n \delta \left[ \tilde{\phi}(x) - 2\pi n \right] \]  

and, using the Poisson summation formula, this becomes
\[
\rho(x) = \frac{\nabla \tilde{\phi}}{2\pi} \sum_m e^{im\tilde{\phi}(x)}
\]

It is convenient to introduce \( \tilde{\phi}(x) \), which is the deviation of \( \tilde{\phi} \) from \( 2\pi x/d \) (i.e. the deviation from a perfect lattice):

\[
\tilde{\phi}(x) = 2\pi \rho_0 x - 2\phi(x)
\]

In terms of \( \phi(x) \), the density operator is

\[
\rho(x) = \left( \rho_0 - \frac{\nabla \phi}{\pi} \right) \sum_m e^{i2m[\pi\rho_0 x - \phi(x)]}
\]

II. RELATION TO DISPLACEMENT AND THE MOMENTUM DENSITY OPERATOR

The field \( \tilde{\phi}(x) \) can be written as

\[
\tilde{\phi}(x) = 2\pi n + (x - x_n) \nabla \tilde{\phi}(x) + \ldots
\]

where \( x_n \) is the position of the particle which is closest to \( x \). Substituting Eq. 6, and neglecting higher order terms in \( x - x_n \) (assuming that \( \tilde{\phi} \) is a smooth function), we get

\[
2\pi \rho_0 x - 2\phi(x) = 2\pi n + (x - x_n) [2\pi \rho_0 - 2\nabla \phi(x)]
\]

\[
\phi(x) \approx \pi \rho_0 [x_n - nd] \equiv \pi \rho_0 u_n
\]

In the last line, we have assumed that \( \nabla \phi(x) \ll \pi \rho_0 \), and neglected the \( \nabla \phi(x) \) term in the right hand side. This is justified for an “almost ordered” configuration, in which the deviation from a perfect lattice is small.

Eq. 9 shows that the field \( \phi(x) \) has a simple interpretation: it is proportional to the local displacement, \( u_n = x_n - nd \), of the \( n \)th particle from its crystalline position, \( nd \). This allows us to identify the momentum density operator, \( \Pi(x) \), which is the canonical conjugate of the field \( \phi(x) \), in terms of the physical momentum \( P_n \). \( \Pi(x) \) satisfies the commutation relation

\[
[\phi(x), \Pi(x')] = i\delta(x - x').
\]

Comparing this to the canonical commutation relation satisfied by \( u_n \) and \( P_n \):

\[
[u_n, P_{n'}] = i\delta_{n,n'}
\]

and using the relation \( \phi(x_n) = \pi \rho_0 u_n \), we identify

\[
\Pi(x) = \frac{P_n}{\pi}
\]
III. HAMILTONIAN IN CONTINUUM FORM

Next, we’d like to write the Hamiltonian in terms of the continuum fields $\phi(x)$ and $\Pi(x)$. Using Eq. 12, it is easy to rewrite the kinetic part of the Hamiltonian:

$$\sum_n \frac{P_n^2}{2m} = \int dx \rho_0 \frac{[\pi \Pi(x)]^2}{2m}. \quad (13)$$

Next, we need to write the potential energy part. In order to do so, we note that for small, slowly varying displacement $\phi(x)$, we expect the potential energy density to be proportional to $(\nabla \phi)^2$. This follows from the invariance of the system under translation, and from the fact that $\nabla \phi = 0$ corresponds to the perfect crystal which is the minimum of the potential energy. The coefficient is given by the bulk modulus of the perfect crystal. Consider a uniform compression of the crystal, for which the displacement field is

$$u_n = -n \delta u \quad (14)$$

The second derivative of the potential energy with respect to $\delta u$ is

$$B \equiv \frac{1}{L} \frac{\partial^2}{\partial \delta u^2} V(u_1, \ldots, u_N) \bigg|_{u_n=0} = \frac{1}{L} \sum_{n,m} \frac{\partial^2 V}{\partial u_n \partial u_m} mn = -\frac{1}{2L} \sum_{n,m} \frac{\partial^2 V}{\partial u_n \partial u_m} (m - n)^2 \quad (15)$$

where $L$ is the length of the system, and in the last equality, we have used the fact that

$$m^2 \sum_n \frac{\partial^2 V}{\partial u_n \partial u_m} = 0, \quad (16)$$

which follows from the translational invariance of $V$. Note that the stability of the crystal requires that $B > 0$. The potential energy for small $\delta u$ is

$$V(\delta u, 2\delta u, \ldots, N\delta u) = V(0, \ldots, 0) + \frac{L}{2} B \delta u^2 \quad (17)$$

Substituting $\delta u \approx \frac{1}{\pi \rho_0} \nabla \phi$, and dropping the constant $V(0, \ldots, 0)$, we get that

$$V \approx \int dx \frac{B}{2(\pi \rho_0^2)^2} (\nabla \phi)^2 \quad (18)$$

Combining Eq. 18 with Eq. 13, we may write the Hamiltonian as

$$H = \frac{u}{2\pi} \int dx \left[ K (\pi \Pi)^2 + \frac{1}{K} (\nabla \phi)^2 \right] \quad (19)$$

where we have defined the sound velocity $u$ and the Luttinger parameter $K$ by

$$u = \sqrt{\frac{B}{m \rho_0}}, \quad (20)$$

$$K = \pi \rho_0 \sqrt{\frac{\rho_0^3}{mB}}. \quad (21)$$

Eq. 19 is the Luttinger liquid Hamiltonian, which describes the low-energy, long-distance behavior of a vast variety of one-dimensional systems. Different systems (bosons/fermions, weak/strong coupling, etc.) are described by different values of $u$ and $K$. 
IV. DIAGONALIZATION OF $H$

The Luttinger liquid Hamiltonian describes a collection of harmonic oscillators, therefore it can be diagonalized exactly. Consider a system of length $L$ with periodic boundary conditions. We transform to Fourier space by writing

$$\phi_p = \frac{1}{\sqrt{L}} \int dx e^{-ipx} \phi(x)$$
$$\Pi_p = \frac{1}{\sqrt{L}} \int dx e^{-ipx} \Pi(x).$$

These fields satisfy

$$[\phi_p, \Pi_{p'}] = i\delta_{p+p'}.$$

The Hamiltonian becomes

$$H = \sum_p \frac{u|p|}{2} \left[ \pi K \Pi_p \Pi_p + \frac{|p|}{\pi K} \phi_p \phi_p \right].$$

We define raising and lowering operators

$$\phi_p = \sqrt{\frac{\pi K}{2|p|}} \left( a_p + a_p^\dagger \right),$$
$$\Pi_p = \frac{1}{i} \sqrt{\frac{|p|}{2\pi K}} \left( a_p - a_p^\dagger \right).$$

The Hamiltonian becomes

$$H = \frac{\pi uK}{2} (\Pi_0)^2 + \sum_p u|p| \left( a_p^\dagger a_p + \frac{1}{2} \right).$$

The first term here describes the center-of-mass kinetic energy of the entire system. The second term describes the energy of the eigenmodes (“phonons”), whose dispersion is linear with momentum, $\varepsilon_p = u|p|$.

V. DENSITY-DENSITY CORRELATION FUNCTION

As an exercise, let us compute density-density correlation in the ground state. The local deviation of the density from its average value is

$$\delta \rho(x) = \frac{\nabla \phi}{\pi} + \rho_0 \cos(2\pi \rho_0 x - 2\phi) + \ldots$$

and we want to compute

$$\langle \delta \rho(x) \delta \rho(0) \rangle$$

where $\langle \cdot \rangle$ represents the expectation value in the ground state.

where the infinite series contains also higher harmonics of the form $\cos[2m(\pi \rho_0 x - \phi)]$, with $m > 1$, which we ignore for now. (These terms will give subleading corrections to the correlation function, which decay faster at long distances than the terms that we will compute.)

Since $\delta \rho(x)$ depends on $\phi(x)$, it will be sufficient to calculate the correlations of $\phi(x)$. In particular, let us calculate
\[
\langle [\phi(x) - \phi(0)]^2 \rangle = \frac{1}{L} \sum_{p,p'} \langle \phi_p \phi_{p'} (1 - e^{ipx}) (1 - e^{ip'x}) \rangle
\]
\[
= \frac{1}{L} \sum_{p,p'} \frac{\pi K}{2\sqrt{|p||p'|}} \langle (a_p + a^\dagger_{-p}) (a_{p'} + a^\dagger_{-p'}) (1 - e^{ipx}) (1 - e^{ip'x}) \rangle
\]
\[
= \frac{\pi K}{L} \sum_p \frac{1 - e^{ipx}}{|p|}
\]
\[
= \pi K \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1 - e^{ipx}}{|p|} e^{-\alpha|p|},
\]  
(29)

where \( \alpha \sim d \) is the large-momentum (short distance) cutoff of the theory. It represents the fact that there can be no lattice vibrations (or phonons) with a wavelength which is shorter than the inter-particle spacing \( d \). (This scheme of introducing a cutoff is called a “soft cutoff”, since the integral over \( p \) is still taken from \(-\infty\) to \( \infty \), but the contributions from high momenta are suppressed. A “hard cutoff” scheme, in which the integral is taken from \(-1/\alpha\) to \(1/\alpha\), is also possible, and leads to the same long-distance behavior.) Evaluating the integral gives

\[
\langle [\phi(x) - \phi(0)]^2 \rangle = \frac{K}{2} \ln \frac{\alpha^2}{x^2 + \alpha^2}
\]
\[
\approx -K \ln \frac{x}{\alpha}
\]  
(30)

where the last equality holds for \( x \gg \alpha \). We can now use this to calculate the density-density correlation function, as follows.

\[
\frac{1}{\pi^2} \langle \nabla \phi(x) \nabla \phi(0) \rangle = -\frac{1}{2\pi^2} \nabla_x \nabla_{x'} \left[ \langle [\phi(x) - \phi(x')]^2 \rangle \right]_{x' = 0}
\]
\[
= -\frac{1}{2\pi^2} \nabla_x \nabla_{x'} \left( -K \ln \frac{x - x'}{\alpha} \right)_{x' = 0}
\]
\[
= \frac{K}{2\pi^2} \frac{1}{x^2}. \tag{31}
\]

To compute the correlations of the cosine term in 27, we write it as

\[
\cos (2\pi \rho_0 x - 2\phi) = \frac{1}{2} \left[ e^{i(2\pi \rho_0 x - 2\phi)} + e^{-i(2\pi \rho_0 x - 2\phi)} \right]. \tag{32}
\]

The only terms which contribute to the correlations of the cosine are

\[
\langle \cos [2\pi \rho_0 x - 2\phi(x)] \cos [-2\phi(0)] \rangle = \frac{1}{4} e^{i2\pi \rho_0 x} \left[ e^{-i2\phi(x)} e^{i2\phi(0)} \right] + c.c. \tag{33}
\]

The term \( e^{-i2\phi(x)} e^{-i2\phi(0)} \) vanishes, since it is not invariant under \( \phi(x) \to \phi(x) + \text{const.} \), which leaves the Hamiltonian invariant. Similarly, the cross term \( \langle e^{-i2\phi(x)} \nabla \phi(0) \rangle \) vanishes.

Now, we need to compute

\[
\langle e^{-2i[\phi(x) - \phi(0)]} \rangle.
\]

This term can be computed as follows. A general theorem says that if \( A \) is an operator which is a linear function of \( a \) and \( a^\dagger \) of a harmonic oscillator (such as \( \phi \)), then
\[ \langle e^A \rangle = e^{\frac{i}{2} \langle A^2 \rangle} \]  

(34)

To prove this, write \( A = ua + va^\dagger \). Since the commutator of \( a \) and \( a^\dagger \) is a c-number, we can use the Baker-Hausdorff theorem for two operators \( B \) and \( C \), whose commutator commutes with both \( B \) and \( C \):

\[ e^{B+C} = e^B e^C e^{-\frac{i}{2} [B,C]} . \]  

(35)

Applying this to \( B = ua, C = va^\dagger \) we get:

\[ \langle e^{ua+va^\dagger} \rangle = \langle e^{va^\dagger} e^{ua} e^{-\frac{i}{2} [a^\dagger, a]} \rangle = \langle e^{va^\dagger} e^{ua} \rangle e^{\frac{i}{2} \langle [a^\dagger, a] \rangle} = e^{\frac{i}{2} \langle (ua+va^\dagger)^2 \rangle} . \]  

(36)

We have used the fact that

\[ \langle e^{va^\dagger} e^{ua} \rangle = \left\langle \left( 1 + va^\dagger + \frac{1}{2} v^2 a^2 + \ldots \right) \left( 1 + ua + \frac{1}{2} u^2 a^2 + \ldots \right) \right\rangle = 1 . \]  

(37)

Therefore

\[ \langle e^{-2i[\phi(x)-\phi(0)]} \rangle = e^{-2(\langle \phi(x)-\phi(0) \rangle)^2} = e^{-2K \ln \frac{x}{\alpha}} \langle \frac{\alpha}{x} \rangle^{2K} . \]  

(38)

Inserting Eq. 31,38 into the correlation function of \( \delta \rho \), we get

\[ \langle \delta \rho (x) \delta \rho (0) \rangle = \frac{K}{2\pi^2 x^2} + \frac{\alpha^2}{2} \left( \frac{\alpha}{x} \right)^{2K} \cos (2\pi \rho_0 x) + \ldots \]  

(39)

where we have truncated higher-order terms which comes from the higher harmonics in Eq. 27. Those terms can be shown to decay with larger exponents than the leading term, and therefore it is justified to ignore them at long distances. The fact that the correlations decay at long distances is a manifestation of the Mermin-Wagner theorem, which says that a continuous symmetry (in our case, translational symmetry) cannot be broken in a quantum one-dimensional system, even at zero temperature. Here, we see that the long-range crystalline correlations decay with a power law of \( 2K \) at large distances, which means that they not long ranged unless \( K = 0 \) (which corresponds, e.g., to the case of particles with infinite mass, \( m = \infty \); see Eq. 21). The power law behavior of \( \langle \delta \rho (x) \delta \rho (0) \rangle \), with a power law that depends continuously on \( K \), is a characteristic behavior of a Luttinger liquid.

VI. SINGLE PARTICLE OPERATORS

So far, there has been no difference between fermions and bosons. To know whether the system is made out of fermions or bosons, one needs to look at correlations of the single particle operator, or the single particle Green’s function,

\[ G_{B/F} (x) = \langle \Psi_{B/F} (x) \Psi_{B/F}^\dagger (0) \rangle \]  

(40)

where \( B \) and \( F \) stand for bosons or fermions.

A. Bosons

To compute this, let us first construct \( \Psi_B (x) \) in terms of the continuum fields \( \phi (x), \Pi (x) \). \( \Psi_B (x) \) should satisfy

\[ \left[ \Psi_B (x), \Psi_B^\dagger (x') \right] = \delta (x - x') . \]  

(41)
We may write

\[ \Psi_B (x) = e^{i\theta(x)}\sqrt{\rho(x)}, \]  

(42)

where \( \theta(x) \) is an unknown operator. To find \( \theta(x) \), we insert Eq. 42 into 41 and find that

\[ e^{i\theta(x)} \sqrt{\rho(x)} \sqrt{\rho(x')e^{-i\theta(x')}} - \sqrt{\rho(x')e^{-i\theta(x')}} e^{i\theta(x)} \sqrt{\rho(x)} = \delta(x - x'). \]  

(43)

For \( x \neq x' \), this is satisfied if \( [\theta(x), \theta(x')] = 0 \). For \( x = x' \), we find that

\[ e^{i\theta(x)} \rho(x) e^{-i\theta(x)} - \rho(x) = \text{const}. \]  

(44)

So the “phase” operator \( \theta(x) \) is proportional to the canonical conjugate of \( \rho(x) \). Looking at Eq. 7, one can show that this is satisfied if

\[ e^{i\theta(x)} \nabla \phi(x') \frac{1}{\pi} e^{-i\theta(x')} = \nabla \phi(x') \frac{1}{\pi} + \delta(x - x') \]  

(45)

which is satisfied for

\[ \left[ \nabla \phi(x') \frac{1}{\pi}, \theta(x) \right] = -i\delta(x - x'). \]  

(46)

Comparing this to Eq. 10, we see that

\[ \frac{1}{\pi} \nabla \theta(x) = \Pi(x), \]  

(47)

or

\[ \theta(x) = \pi \int_{-\infty}^{x} dx' \Pi(x'). \]  

(48)

We conclude that the bosonic field operator is represented as

\[ \Psi_B(x) = e^{i\theta(x)}\sqrt{\rho(x)} = \sqrt{\rho_0 e^{i\theta(x)}} + \ldots \]  

(49)

where the sum includes “subleading” terms, which give corrections to the correlations of \( \Psi_B(x) \) which decay faster than the leading term. The correlations of \( \Psi_B(x) \) can be calculated in a similar way to the calculation of the density-density correlations in the previous section, by expanding \( \theta(x) \) in terms of the raising and lowering operators. The result is

\[ G_{B/F} (x) = \left< \Psi_{B/F} (x) \Psi_{B/F}^\dagger (0) \right> = \rho_0 \left< e^{i\theta(x)} e^{-i\theta(0)} \right> + \ldots \]

\[ = \rho_0 e^{-\frac{i}{2} [\theta(x) - \theta(0)]^2} + \ldots \]

\[ = \rho_0 \left( \frac{\alpha}{x} \right) \frac{1}{x} + \ldots \]  

(50)

where, as before, \ldots represented terms which decay with a larger power than \( \frac{1}{x} \). We see that there is no true condensate in this system, i.e. there is no off-diagonal long range order \( (G_{B/F} (x) \rightarrow \text{const as } x \rightarrow \infty) \) for any finite \( K \). This is again a consequence of the Mermin-Wagner theorem, applied to the U(1) charge conservation \( (\Psi_B(x) \rightarrow e^{i\phi} \Psi_B(x)) \), where \( \phi \) is arbitrary). Note also that the free boson behavior, \( G_{B/F} (x) \rightarrow \text{const.} \), is reproduced for \( K \rightarrow \infty \). Although we derived the Luttinger liquid Hamiltonian as a low-energy description of the system for strong coupling, it actually holds much more generally, and describes the system at low energies for any coupling. For very weak coupling, we get \( K \gg 1 \), and for strong coupling (the regime we worked at so far) \( K \ll 1 \).
B. Fermions

Next, we need to construct the fermionic field operator \( \Psi_F(x) \). Unlike the bosonic field operator, \( \Psi_F \) at different points anticommute:

\[
\{ \Psi_F(x), \Psi_F(x') \} = 0 \\
\{ \Psi_F(x), \Psi_F^+(x') \} = \delta(x - x').
\]  

(51)

To get this, we define the “string” operator which gives the parity of the number of particles which is to the left of a point \( x \):

\[
e^{i\pi N(x)} = e^{\pm i\pi \int_{-\infty}^x dx' \frac{1}{\pi} \nabla \phi(x')} = e^{\pm \frac{1}{2} \phi(x)}
\]  

(52)

where we have used the definition of the counting field \( \tilde{\phi}(x) \) (see Eq. 4).

\[
\rho(x) \sim \frac{1}{\pi} \nabla \phi(x) + \text{(oscillatory terms in } x \text{ which vanish upon integration)}
\]  

(53)

(see Eq. 7), and assumed that \( \phi(-\infty) = 0 \). Next, note that

\[
\Psi_F(x) e^{i\pi N(x')} = \text{sign}(x' - x) e^{i\pi N(x')} \Psi_F(x).
\]  

(54)

To prove this, simply note that \( \Psi_F(x) \) changes the fermion number by 1. Therefore moving it through \( e^{i\pi N(x')} \) gives a minus sign if \( x' > x \), and no sign change if \( x' \leq x \). With this observation, one can guess that the fermion operator should be of the form

\[
\Psi_F \sim \Psi_B e^{i\pi N(x)} \sim \sqrt{\rho_0} e^{i[\Theta(x) + \frac{1}{2} \phi(x) + \frac{1}{2} \phi'(0)]} + \ldots
\]  

\[
= \sqrt{\rho_0} \left( e^{i[\Theta(x) - \phi(x)] + ik_F x} + e^{i[\Theta(x) + \phi(x)] - ik_F x} \right) + \ldots
\]  

(55)

where \( k_F = \pi \rho_0 \) is the Fermi momentum of free electrons with the same density. Interestingly, we got that the field operator \( \Psi_F(x) \) has components with oscillate at a wavevector of \( \pm k_F \), precisely the same wavevectors of the field operator of free electrons close to the Fermi energy. From Eq. 55, one can calculate the fermionic single particle Green’s function:

\[
\langle \Psi_F(x) \Psi_F^+(0) \rangle \sim \rho_0 \left( e^{i[\Theta(x) - \phi(x)]} e^{-i[\Theta(0) - \phi(0)]} \right) e^{ik_F x}
\]  

\[
\sim \rho_0 \left( \frac{\alpha}{x} \right)^{\frac{1}{2}} (K + \frac{x}{2}) e^{ik_F x}.
\]  

(56)

So, the leading (slowest decaying) single-particle fermionic correlations decay as \( x^{-\frac{1}{2}} (K + \frac{x}{2}) \) and oscillate with a wavevector \( k_F \). Note that the exponent is always greater or equal to 1, with \( K = 1 \) corresponding to correlations which decay as \( \frac{1}{x} \). Interestingly, this is precisely the power law with which single-particle correlations decay in a one-dimensional free electron gas:

\[
\langle \Psi_F(x) \Psi_F^+(0) \rangle_{\text{Free}} \sim \sum_k e^{ikx} \langle \tilde{\psi}_k \tilde{\psi}_k^+ \rangle
\]  

\[
\sim \sum_k e^{ikx} \left[ \Theta(k - k_F) + \Theta(-k - k_F) \right]
\]  

\[
\sim \frac{\sin(k_F x)}{x}
\]  

(57)

where \( \Theta(k) \) is the Heaviside step function. Again, although our derivation was strictly applicable for the strongly interacting (\( K \gg 1 \)) case, the results actually apply for any strength of the interactions. By comparing the form of the long-range part of the single particle Green’s function to the exact expression in the non-interacting case, we conclude that \( K = 1 \) corresponds to non-interacting electrons.
VII. REFERENCES

This lecture is based on the third chapter of Thierry Giamarchi’s book, “Quantum physics in one dimension”.

There are numerous excellent reviews of bosonization and Luttinger liquid theory, which emphasize different aspects of it. In particular, two reviews which I found particularly useful (especially for systematically deriving bosonization starting from the weak-coupling limit) are:
