Strongly correlated systems
in atomic and condensed matter physics

Lecture notes for Physics 284

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Chapter 12

Collective modes in interacting Fermi systems

12.1 Collective modes and response functions. RPA approach in the normal state.

12.1.1 From response functions to collective modes

Consider response function $\chi_A(q,\omega)$ for some quantity $A(q,\omega)$. It is defined as the expectation value of the operator $\hat{A}$ induced by the external field $h_A(q,\omega)$, which is thermodynamically conjugate to it. Here thermodynamically conjugate means that the field $h_A(q,\omega)$ contributes a term to the Hamiltonian

$$\mathcal{H}_{\text{external}} = h_{Aq\omega} e^{i\omega t} \hat{A}_{-q} + \text{c.c.} \tag{12.1}$$

Hence the response function is defined by

$$\langle A(q,\omega)\rangle = \chi_A(q,\omega) h_A(q,\omega) \tag{12.2}$$

Generally to get a finite expectation value of $\langle A(q,\omega)\rangle$ one needs to have a nonzero external field $h_A(q,\omega)$. One important exception is when the response function $\chi_A(q,\omega)$ is infinite. Then it may be possible to have a finite $\langle A(q,\omega)\rangle$ with $h_A(q,\omega)=0$. A more accurate argument shows that collective excitations in the system correspond to poles of the response function $\chi_A(q,\omega)$ [1, 8].

12.1.2 Equations of motion method

With few exceptions calculating response functions for interacting many-body systems can not be done exactly. The approach that we discuss in this chapter is based on writing equations of motion for density and spin operators $\dot{O} = \frac{1}{i}[\mathcal{H}, O]$, and then doing certain approximations to bring them to a closed form. When O is a two particle operator of the type $c^\dagger_{k\sigma} c_{k'\sigma'}$, its commutator with the interaction term produces a four fermion term, so equations of motion do not close. RPA approximation replaces the true interacting Hamiltonian with an effective non-interacting Hamiltonian, where interactions are replaced by some effective field[8, 2]. For example, when the system is perturbed by the potential at wavevector q and frequency ω, we take the effective potential to be $V_{\rm eff}(q\omega) = V(q)\langle\rho(q,\omega)\rangle\, e^{i\omega t}$, where $V(q)$ is the interaction strength and $\langle\rho(q,\omega)\rangle$ is the induced density. This potential can be understood as an effective time- and space- dependent Hartree potential created by other fermions. So fermions are scattered by the potential, which is the sum of the external and polarization potentials.

12.1.3 Density sector. Zero sound mode

We consider an interacting Fermi system in the normal state

$$\mathcal{H} = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} + V \sum_k c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{-k'+q\uparrow} \tag{12.3}$$

To calculate the density-density response function we assume that the system is perturbed by the external potential at wavevector q and frequency ω

$$\mathcal{H}_{\rm probe} = \frac{h_{q\omega}}{2} e^{i\omega t} \sum_{p\sigma} c^\dagger_{p-q\sigma} c_{p\sigma} + \text{c.c.} \tag{12.4}$$

We define operators

$$\begin{aligned} \rho_{kq\sigma} &= c^\dagger_{k+q\sigma} c_{k\sigma} \\ \rho_{kq} &= \rho_{kq\uparrow} + \rho_{kq\downarrow} \end{aligned} \tag{12.5}$$

and write their equations of motion. In writing equations of motion we assume that the system develops expectation values of the density at wavevector q, which we will need to determine self-consistently. We take

$$\mathcal{H}_{\rm eff} = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} \sum_k c^\dagger_{k-q\sigma} c_{k\sigma} \tag{12.6}$$

and find

$$\frac{1}{i}\dot{\rho}_{kq\sigma} = [\mathcal{H}_{\rm eff}, \rho_{kq\sigma}] = (\xi_{k+q} - \xi_k)\rho_{kq\sigma} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} \left(n_k - n_{k+q}\right) \tag{12.7}$$

In the last term $n_{k\sigma} = c^\dagger_{k\sigma} c_{k\sigma}$ and $n_{k+q\sigma} = c^\dagger_{k+q\sigma} c_{k+q\sigma}$ are, in principle, operators, but we replace them by their expectation values in the ground state. We take $\langle \rho_{kq}(t) \rangle = \langle \rho_{kq\omega} \rangle e^{i\omega t}$ and find

$$\left(\omega - (\xi_{k+q} - \xi_k) \right) \rho_{kq\omega} = (V\langle \rho_{q\omega} \rangle + h_{q\omega})(n_k - n_{k+q}) \tag{12.8}$$

We have $\langle \rho_{q\omega} \rangle = \sum_k \langle \, \rho_{kq(\omega)} \, \rangle$ and find

$$\langle \rho_{q\omega} \rangle = \chi_0(q,\omega)(V \langle \rho_{q\omega} \rangle + h_{q\omega}) \tag{12.9}$$

where

$$\chi_0(q,\omega) = \sum_k \frac{n_k - n_{k+q}}{\omega - (\xi_{k+q} - \xi_k)} \tag{12.10}$$

Thus we find RPA expression for the response function

$$\chi_{\rm RPA}(q,\omega) = \frac{\chi_0(q,\omega)}{1 - V\chi_0(q,\omega)} \tag{12.11}$$

Poles of the density-density response function describe collective excitations in the system. So we look for the solution of equation

$$1 - V\chi_0(q,\omega) = 0 \tag{12.12}$$

In the longwavelength limit ($q \to 0$) and at $T = 0$ we can take

$$\chi_0(q,\omega) = \sum_k \frac{(-\frac{\partial n}{\partial \epsilon}\,)\,(\,\vec{q}\vec{v}_p\,)}{(\,\omega - \vec{q}\vec{v}_p\,)} = \frac{N(0)}{2} \int \frac{d\Omega_p}{4\pi} \, \frac{\vec{q}\vec{v}_p}{(\,\omega - \vec{q}\vec{v}_p\,)} = N(0)[\,-1 + \frac{\lambda}{2}\log\frac{\lambda+1}{\lambda-1}\,] \tag{12.13}$$

where $\lambda = \omega/qv_{\rm f}$. In writing equation (12.13) we used $\frac{\partial n}{\partial \epsilon} = -\delta(\epsilon - \epsilon_{\rm f})$. So we have

$$\frac{1}{VN(0)} = -1 + \frac{\lambda}{2}\log\frac{\lambda+1}{\lambda-1} \tag{12.14}$$

For repulsive interactions this equation describes zero sound excitations[8, 1]. Note that zero sound mode is above the particle-hole continuum.

12.1.4 Spin sector. Stoner instability

We now consider probing the system in the spin sector

$$\mathcal{H}_{\rm probe} = \frac{h^Z_{q\omega}}{2} e^{i\omega t} \sum_{p\sigma} \sigma c^\dagger_{p-q\sigma} c_{p\sigma} + \text{c.c.} \tag{12.15}$$

Analysis similar to the one we discussed in the previous subsection gives

$$\chi^Z_{\rm RPA}(q,\omega) = \frac{\chi_0(q,\omega)}{1 + V\chi_0(q,\omega)} \tag{12.16}$$

Spin wave modes can be found from the equation

$$-\frac{1}{VN(0)} = -1 + \frac{\lambda^z}{2}\log\frac{\lambda^z+1}{\lambda^z-1} \tag{12.17}$$

For repulsive interactions with $VN(0) > 1$ we find imaginary frequency of the modes. This corresponds to exponentially growing magnetization and signals instability of the system to spontaneous polarization[8, 1]. This is the so-called Stoner instability[12].

12.2 Anderson-Bogoliubov mode in the paired state

12.2.1 RPA approach in the paired state.

In the presence of pairing we need to change the "bare" quadratic Hamiltonian to the BCS form

$$\mathcal{H} = \sum_{k\sigma} \left\{ \xi_k c^{\dagger}_{k\sigma} c_{k\sigma} + (\,\Delta\, c^{\dagger}_{k\uparrow}\, c^{\dagger}_{-k\downarrow} + \text{c.c.}\,) \right\} \tag{12.18}$$

Moreover, external field that couples to the density operator induces not only the expectation value of the density $\rho_q = \sum_{k\sigma} c^{\dagger}_{k+q\sigma} c_{k\sigma}$, but also anomalous expectation values

$$\begin{aligned} \langle b^{\dagger}_q \rangle &= \sum_k \langle c^{\dagger}_{k+q\uparrow} c^{\dagger}_{-k\downarrow} \rangle \\ \langle b_{-q} \rangle &= \sum_k \langle c_{-k-q\downarrow}\, c_{k\uparrow} \rangle \end{aligned} \tag{12.19}$$

This reflects particle-hole mixing in the superfluid state arising from the condensate of Cooper pairs Hence

$$\begin{aligned} \mathcal{H}_{\text{eff}} &= \sum_{k\sigma} \left\{ \xi_k c^{\dagger}_{k\sigma} c_{k\sigma} + (\,\Delta\, c^{\dagger}_{k\uparrow}\, c^{\dagger}_{-k\downarrow} + \text{c.c.}\,) \right\} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} \sum_k c^{\dagger}_{k-q\sigma} c_{k\sigma} \\ &+ V \langle b^{\dagger}_q \rangle \sum_k c_{-k\downarrow} c_{-k+q\uparrow} + V \langle b_{-q} \rangle \sum_k c^{\dagger}_{k-q\uparrow} c^{\dagger}_{-k\downarrow} \end{aligned} \tag{12.20}$$

Equations of motion couple particle-hole operators $\rho_{kq\sigma} = c^{\dagger}_{k+q\sigma} c_{k\sigma}$ with two-particle-particle operators $b^{\dagger}_{kq} = c^{\dagger}_{k+q\uparrow} c^{\dagger}_{-k\downarrow}$ and two-hole operators $b_{k\,-q} = c^{\dagger}_{-k-q\downarrow} c_{k\uparrow}$. We have

$$\begin{aligned} [\mathcal{H}_{\text{eff}}, \rho_{kq\uparrow}] &= (\xi_{k+q} - \xi_k) \rho_{kq\uparrow} - \Delta b^{\dagger}_{kq} + \Delta^* b_{k\,-q} + \left(\frac{h_{q\omega}}{2} + V \frac{\langle \rho_{q\omega} \rangle}{2} \right) e^{i\omega t} (n_k - n_{k+q}) \\ &+ V \langle b^{\dagger}_q \rangle \langle c_{-k\downarrow} c_{k\uparrow} \rangle + V \langle b_{-q} \rangle \langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle \end{aligned} \tag{12.21}$$

and similar equations for $[\mathcal{H}_{\text{eff}}, \rho_{kq\downarrow}]$, $[\mathcal{H}_{\text{eff}}, b^{\dagger}_{kq}]$ and $[\mathcal{H}_{\text{eff}}, b_{kq}]$. These equations of motion can be solved and in the BCS regime we obtain a collective mode with the long wavelength dispersion [2]

$$\omega_q = \frac{v_{\text{f}}}{\sqrt{3}}\, q \tag{12.22}$$

This is the celebrated Anderson-Bogoliubov mode[3, 2].

12.2.2 Hydrodynamic approach

We now present an alternative approach to calculating the sound mode (12.22). We write hydrodynamic equations of motion[11, 7]

$$\begin{aligned} \frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) &= 0 \\ m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}(\frac{1}{2}mv^2 + \mu(n)) &= 0 \end{aligned} \quad (12.23)$$

In the linearized regime, the second equation can be written as

$$m\frac{\partial \vec{v}}{\partial t} + \left(\frac{\partial n}{\partial \mu}\right)\vec{\nabla}n = 0 \quad (12.24)$$

Hence

$$\frac{\partial^2 n}{\partial t^2} - \frac{n}{m}\left(\frac{\partial n}{\partial \mu}\right)\nabla^2 n = 0 \quad (12.25)$$

In the BCS regime $\mu = E_F = \frac{k_{\rm f}^2}{2m} \sim n^{2/3}$ and $\frac{\partial n}{\partial \mu} = \frac{2}{3}\frac{\mu}{n}$. This gives

$$\omega_q^2 = \frac{k_{\rm f}^2}{3m}q^2 = (\frac{v_{\rm f}}{\sqrt{3}}q)^2 \quad (12.26)$$

We can apply hydrodynamic analysis in the BEC regime using

$$\mu = g_M n_M(1 + \frac{32}{3\sqrt{\pi}}\sqrt{n_M a_M^3} + \dots) \quad (12.27)$$

Here n_M is the density of molecules and g_M is the interaction strength between molecules, which can be expressed using the mass of molecules M and their scattering length a_M as $g_M = \frac{2\hbar^2 a_M}{M}$. Analysis in ref[10] shows that $a_M = 0.6a$. So we find

$$v = \frac{\hbar}{M}\sqrt{2a_M n_M} \quad (12.28)$$

At unitarity we expect to find

$$\mu = (1+\beta)\frac{\hbar^2}{2m}(6\pi^2)^{2/3}n^{2/3} \quad (12.29)$$

where β is known from theory. In the mean-field analysis $\beta = -0.41$, Quantum Monte Carlo gives $\beta = -0.58$.

Hydrodynamic approach can be extended to include effects of the confining potential[13, 14]. Equations of motion in this case can be written as

$$\frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) = 0 \quad (12.30)$$

$$m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}(\frac{mv^2}{2} + \mu(n) + V_{\rm ext}) = 0 \quad (12.31)$$

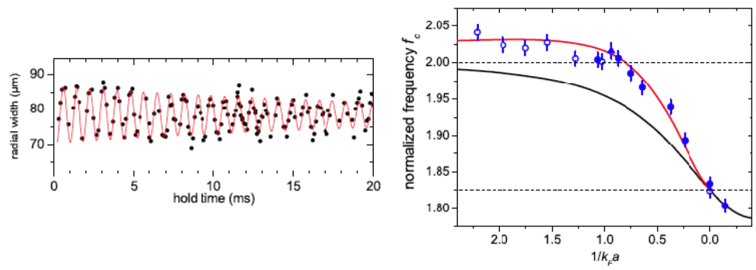

Figure 12.1: Compression oscillations in superfluid Fermi gas. Second figure shows good agreement of theoretical calculations with experiments. Figures taken from [4].

To obtain linearized equations we can take the time derivative of (12.30) and substitute (12.31) for $\partial\vec{v}/\partial t$. In the second term in (12.30), the time derivatie should only be applied to $\vec{v}$, since both $\partial n/\partial t$ and $\vec{v}$ are first order in the deviation from equilibrium. However we can not take n outside $\vec{\nabla}$ since the equilibrium density is not uniform. We find the following linearized equation

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{1}{m}\,\vec{\nabla}(\,n\,\frac{\partial\mu}{\partial n}\,\vec{\nabla}\,\delta n) = 0 \tag{12.32}$$

Far on the BEC side these equations can be simplified further. In this case the system is similar to a gas of weakly interacting bosons, so we can take $\frac{\partial\mu}{\partial n} = g$, where g is a constant. Then we have

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{g}{m}\,\vec{\nabla}(\,n\,\vec{\nabla}\delta n) = 0 \tag{12.33}$$

In equilirium $gn(r) = \mu - V_{\rm ext}(r)$. Hence we find

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{1}{m}\,[\,\vec{\nabla}V_{\rm ext}\,\vec{\nabla}\delta n - (\mu - V(r))\nabla^2\delta n]\;=\;0 \tag{12.34}$$

This equation needs to be solved subject to the boundary conditions that the solution is nonsingular at the origin and vanishes at the Thomas-Fermi radius, where the equilibrium density goes to zero.

Experimentally one often studies lowest energy collective modes. Figure (12.1) shows analysis of the quadrupolar (compressional) mode across the BCS/BEC crossover. It may seem surprising that the mode frequency does not change dramatically with interactions. The sound velocity should depend very strongly on the scattering length on the BEC side and at unitarity. A hand-waving argument is that for stronger interactions the sound wave velocity goes up dramatically, but so does the size of the cloud (think of the Thomas-Fermi approximation). And in the oscillation frequency, which we can estimate as the ratio of the two quantities, the net change ends up being small [9].

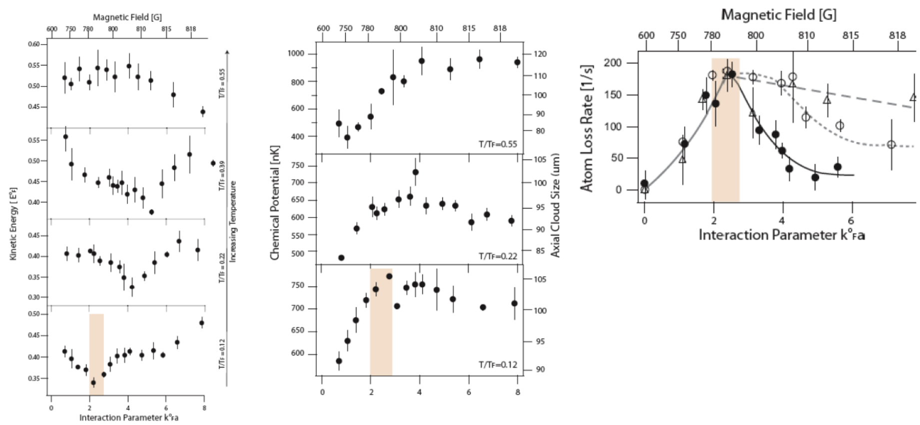

Figure 12.2: Signatures of Stoner instability. Figures taken from [6].

12.3 Competition of Stoner instability and molecule formation near Feshbach resonance

12.3.1 Ferromagnetism in itinerant electron systems

Question of magnetism in itinerant systems of fermions is one of the longest standing problems in physics. Mean-field argument by Stoner suggests that such transition should take place when interactions become strong enough. Using analysis of collective modes we derived earlier that such transition occurs when $N(0)V = 1$. Several counter-arguments have been suggested to Stoner's original mean-field reasoning. For example, Kanamori argued that one should take screened value of the interaction, which may never exceed the Fermi energy. Since the density of states $N(0)$ is inversely proportional to the Fermi energy, Kanamori argued that $N(0)V$ can not be made arbitrarily large, as Stoner's simple argument would suggest.

The idea of studying ferromagnetism with ultracold fermions has been proposed by several groups[**?**].

12.3.2 MIT experiments

Recent experiments on the BEC side of the Feshbach resonance have been interpreted as providing signatures of the Stoner instability [6]. Main features observed in experiments are summarized in figure 12.2. There is an abrupt increase in the kinetic energy as the system approaches the resonance, a maximum in the system size, and suppression of the molecule formation rate.

To get strong repulsive interaction one needs to bring the system to the BEC side of the resonance close to unitarity. The equilibrium state in this case would be a condensate of molecule. In interpreting their experiments as Stoner instability Jo et al. assumed that the rate of molecule formation is smaller than the rate with which magnetization develops. Our goal is to analyze the rate of molecule formation taking into account many-body character of the system.

12.3.3 Molecule formation

We start with interacting fermions in the normal state and analyze the rate of molecule formation in the same approach as we used earlier. We analyze collective modes that correspond to the Cooper channel. If we find imaginary value for the frequency, it will suggest that the system is unstable toward developing pairing with the rate given by the magnitude of the mode frequency[5].

We use RPA analysis

$$\mathcal{H} = \sum_k \xi_k c^\dagger_{k\sigma} c_{k\sigma} + (V b^*_q + h^*_{\Delta q}) \sum_k c_{-k\downarrow} c_{k+q\uparrow} + (V b_q + h_{\Delta q}) \sum_k c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} \tag{12.35}$$

We take the operator $b^\dagger_{pq} = c^\dagger_{p+q\uparrow} c^\dagger_{-p\downarrow}$. Equations of motion for $b^\dagger_{pq}$ are given by

$$[\mathcal{H}, b^\dagger_{pq}] = (\xi_{p+q} + \xi_{-p}) b^\dagger_{pq} + (V b^*_q + h^*_{\Delta q})\,(1 - n_{p+q} - n_{-p}) \tag{12.36}$$

We find for the Cooper pair response functions (Cooperon)

$$\begin{aligned} \chi^C_{RPA} &= \frac{\chi^C_0}{1 - V\chi^C_0} \\ \chi^C_0 &= \sum_p \frac{1 - n_{p+q} - n_{-p}}{\omega - (\xi_{p+q} - \xi_{-p})} \end{aligned} \tag{12.37}$$

To find collective modes we need to solve

$$\frac{1}{V} - \sum_p \frac{1 - n_{p+q} - n_{-p}}{\omega - (\xi_{p+q} - \xi_{-p})} = 0 \tag{12.38}$$

We can again trade the bare interaction for the scattering length using

$$\frac{m}{4\pi a_s} = \frac{1}{V} + \sum_k \frac{m}{k^2} \tag{12.39}$$

For the most unstable mode at $q = 0$ we obtain

$$\frac{m}{4\pi a_s} - \sum_p \left[\frac{1 - 2n_p}{\omega - 2\xi_p} + \frac{m}{p^2} \right] = 0 \tag{12.40}$$

The rate of molecule formation as a function of the scattering length is shown in fig. 12.3. We can also use this analysis to calculate the rate of change of the kinetic energy of fermions that do not become converted into molecules. Essentially the argument is that the imaginary part of the pole of the Cooperon gives the rate of molecule formation and the real part gives the energy of the

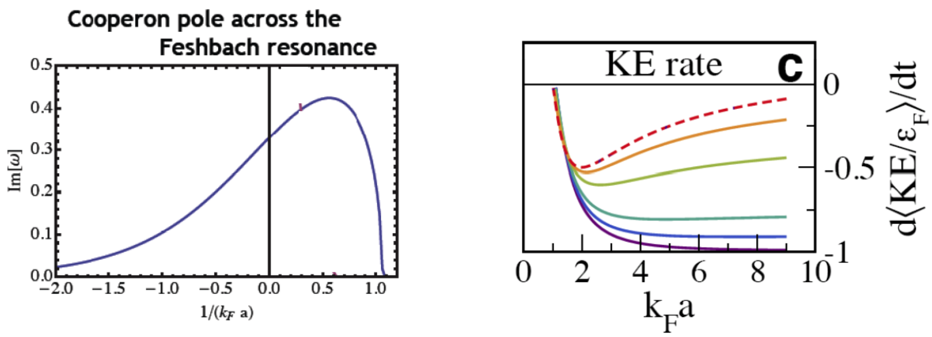

Figure 12.3: The left plot shows the rate of molecule formation. The right plot shows the rate of change of the kinetic energy of fermions due to molecule formation [5].

molecular state, i.e. the energy of fermions being "extracted" from the sea as a result of the molecule formation. Then the product of the two gives the rate of change of the fermion kinetic energy. Both quantities behave qualitatively similar to the experimental results shown in fig. 12.2.

We observe that the physics of molecule formation can explain two features observed in experiments [6]. Additional experiments are need to discriminate between molecule formation and the Stoner instability in the vicinity of the Feshbach resonance.

12.4 Problems to Chapter 12

Problem 1. Scaling solution

In the limit when $\partial\mu/\partial n = g$ is constant and the trap is parabolic, one can solve nonequilibrium hydrodynamic equations beyond the linearized approximation discussed in section 12.2.2. For example, let us consider exciting the system by changing the strength of the periodic potential. Equation (12.31) becomes

$$m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{mv^2}{2} + gn + \frac{1}{2}m\omega^2(t)\,r^2\right) \quad = \quad 0 \qquad (12.41)$$

a) Show that equations (12.30) and (12.41) can be solved by the scaling ansatz

$$\begin{aligned} \vec{v}(r,t) &= \vec{r}\,\frac{\dot{b}(t)}{b(t)} \\ n(r,t) &= \frac{1}{b^3(t)}\,n_0\left(\frac{r}{b(t)}\right) \end{aligned} \qquad (12.42)$$

Here $n_0(r)$ is the equilibrium density when the confining potential strength is

ω_0, and $b(t)$ is the scaling function that satisfies

$$\frac{\ddot{b}}{b} = -\omega^2(t) + \frac{\omega_0^2}{b^4} \tag{12.43}$$

b) Solve the problem of sudden release of the condensate from the trap. This corresponds to taking $\omega(t) = \theta(-t)\,\omega_0$ and initial conditions $b(t = 0) = \dot{b}(t = 0) = 0$.

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