AP 295b **Problem Set 2** **Due: Thursday, October 17, 2002**

Problem 1

In the "jellium model" of a metal, the pressure P is given by $P = -\frac{\partial E}{\partial V}$, where E is the energy of the electron gas in uniform positive background. Express the pressure and the compressibility $\kappa = -V\frac{\partial P}{\partial V}$ in terms of r_s, using the Hartree-Fock approximation.

Problem 2

A set of N fermions has a weak short-range two-body interaction $V(\mid \vec{r} - \vec{r\prime} \mid)$, and is confined to a box of volume Ω. In the Hartree-Fock approximation, at $T = 0$, what is the value of the chemical potential μ and what is the value of the density of states at the Fermi energy?

Problem 3

Derive the equation of motion for $\rho^{+}_{pq\sigma}$ in the generalized RPA:

$$[H, \rho^{+}_{pq\sigma}] = \{\omega^{0}_{pq} + \sum_{k} n^{0}_{k\sigma}(V_{k-p} - V_{k-p-q})\}\rho^{+}_{pq\sigma} - \sum_{k\sigma\prime}(V_q - V_{k-p}\delta_{\sigma\sigma'})(n^{0}_{p+q\sigma} - n^{0}_{p\sigma})C^{0}_{k+q\sigma'}C_{k\sigma'}$$

Problem 4

Consider the Hubbard model on a square lattice

$$\mathcal{H} = -t\sum_{<ij>} C^{+}_{i\sigma}C_{j\sigma} + U\sum_{i} n_{i\uparrow}n_{i\downarrow}$$

Here $< ij >$ stands for the nearest neighbors and the $n_{i\sigma} = C^{+}_{i\sigma}Ci\sigma$

a) Show that C.N. Yang's η-operator

$$\eta^{+} = \sum_{i}(-)^{i}C^{+}_{i\uparrow}C^{+}_{i\downarrow}$$

creates an elementary excitation in the system.

Hint: use the equations of motion method.

b) Show that the charge operator (relative to half-filling)

$$\mathcal{Q} = \frac{1}{2}\sum_{i\sigma}(n_{i\sigma} - \frac{1}{2})$$

and the η-operators (i.e. η and η^+) can be combined to form the pseudospin $SU(2)$ algebra

$$[J_\alpha, J_\beta] = i\varepsilon_{\alpha\beta\delta} J_\delta$$

This algebra can be combined with the spin $SU(2)$ to give the $SO(4)$ symmetry of the Hubbard model at half-filling (C.N. Yang and S.C. Zhang, 1990)

Problem 5

Consider a 2-dimensional electron gas in a uniform perpendicular magnetic field. For spin polarized electrons, the non-interacting Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2m}\left(-i\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2$$

When the vector potential is in the Landau gauge $(A_x = A_z = 0, A_y = B \cdot x)$ the Schroedinger equation takes the form

$$\frac{-1}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2m}\left(-i\frac{\partial}{\partial y} - \frac{e}{c}B \cdot x\right)^2 \psi = E \cdot \psi$$

The solutions of this equation are

$$\phi_{nk}\vec{r} = \frac{e^{-iky}}{\sqrt{L_y}}\phi_n(x - k\ell_0^2), n = 0, 1, ...$$

$$\phi_n(x) = \frac{1}{(\sqrt{\pi}2^n n!\ell_0)^{\frac{1}{2}}} \cdot exp\left[-\frac{x^2}{2\ell_0^2}\right] H_n\left(\frac{x}{\ell_0}\right)$$

Here $l_0 = \frac{c}{eB}$ (remember that $\hbar = 1$) and L_y is the length of the system in the y-direction. The energy eigenvalues, $E = \omega_c\left(n + \frac{1}{2}\right)$ with $\omega_c = \frac{eB}{mc}$, do not depend on k. Hence, we have macroscopic degeneracy on each level $N_\phi = \frac{L_x L_y}{2\pi\ell_0^2}$ (assuming periodic boundary conditions, which implies quantization of k in units of $\frac{2\pi}{L_y}$). Discreet energy levels are called Landau levels (for a more detailed discussion see e.g. Landau and Lifshitz, v.3). Macroscopic degeneracy of electrons on Landau levels amplifies the importance of the Coulomb interaction and leads to a large variety of interesting electron states. In this problem we will consider the possibility of formation of unidirectional CDW states. Such states have been recently observed experimentally (see e.g. PRL 82, 394 (99)).

When the cyclotron frequency ω_c is large, one can neglect all the orbitals except the lowest one (this assumes that the total number of electrons $N = \nu^* N_\phi$, with $\nu^* < 1$). With such projection into the lowest Landau level the electron operators are given by

$$\psi(\vec{r}) = \sum_k C_{0,k}\phi_{0,k}(\vec{r})$$

a) Show that after projecting into the lowest Landau level the density operator $p(\vec{q}) = \int d^2re^{-i\vec{q}\vec{r}}\rho(\vec{r})$ takes the form

$$\rho(\vec{q}) = \sum_k C^+_{0,k+\frac{q_y}{2}} C_{0,k-\frac{q_y}{2}} e^{-ikq_x\ell_0^2 - \frac{1}{4}q^2\ell_0^2}$$

b) Explain why the Coulomb Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2L_xL_y}\sum_{\vec{q}} V_c(\vec{q})\rho(\vec{q})\rho(-\vec{q}) + const$$

where

$$V_c(\vec{q}) = \frac{2\pi^2}{\varepsilon\cdot \mid \vec{q} \mid}$$

c) Consider a variational state shown in the figure at the end of the problem set. Electrons form alternating filled and empty domains in k-space, with width $\Delta k_1 = \frac{a\nu^*}{\ell_0^2}$ and $\Delta k_2 = \frac{a(1-\nu^*)}{\ell_0^2}$ respectively (remember that k values are quantized as $k_n = \frac{2\pi}{L_y}n$). Discuss the kind of structure that this state has in real space.

Hint: think of the guiding center coordinate $x = k\ell_0^2$ in the expression for $\phi_{0k}(\vec{r})$.

d) Explain why this state is a Slater determinant state.

e) Calculate $\rho(\vec{q})$ for this state.

f) Use results of part e) to show that the Hartree energy per one electron is

$$\varepsilon_{Hartree} = \frac{1}{2\nu^*}\sum_{G_n} \Delta_n^2 U_{Hartree}(G_n\cdot \hat{e}_x)$$

where $G_n = \frac{2\pi}{a}n, n\epsilon Z$

$$U_{Hartree}(\vec{q}) = \frac{1}{2\pi\ell_0^2}V_c(\vec{q})e^{\frac{-q^2\ell^2}{2}}$$

$$\Delta_n = \frac{1}{\pi n} \sin(\pi n \nu^*)$$

Hint: remember that the Hartree energy is

$$E_{Hartree} = \frac{1}{2L_x L_y} \sum_q V_c(\vec{q}) < \rho(\vec{q}) >< \rho(-\vec{q}) >$$

g) Show that the exchange energy per particle is

$$E_{Fock} = \frac{1}{2v^*} \sum_{G_n} \Delta_n^2 U_{Fock}(G_n \cdot \hat{e}_x)$$

$$U_{Fock}(\vec{q}) = -\int \frac{d^2p}{(2\pi)^2} e^{\frac{-p^2\ell^2}{2}} e^{i(p_x q_y - p_y q_x)\ell_0^2} V_c(\vec{p})$$

h*) Minimize the total energy with respect to a at $\nu^* = \frac{1}{2}$

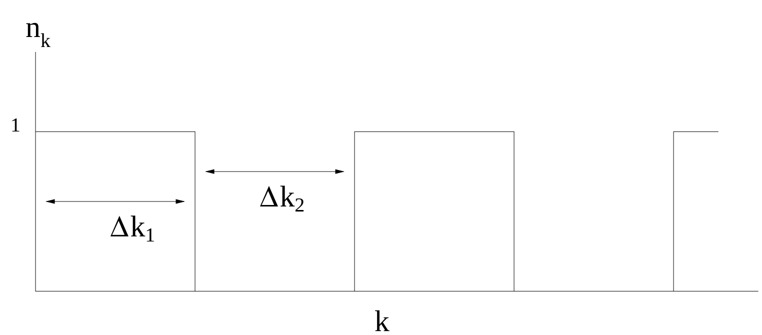

FIG. 1: Problem 5