AP 295b **Problem Set 5** **Due: Tuesday, December 3, 2002**

Problem 1

Show that the particle current density in the absence of magnetic field may be written as, on taking account of the 2 spin orientations

$$\vec{J}(\vec{r}t) = -\frac{1}{m} \lim_{\substack{\vec{r}\to\vec{r}' \\ t\to t'+0}} (\vec{\nabla}_{\vec{r}'} - \vec{\nabla}_{\vec{r}})\, G\,(\vec{r}'t', \vec{r}t)$$

Problem 2

When discussing non-equilibrium phenomena in solid state systems, one often needs to define the following Green's functions:

Retarded

$$G_{AB}^{Ret}\,(t) = i\, < 0|\hat{A}(t)\,\hat{B}(0) - \hat{B}(0)\,\hat{A}(t)|0 > \; \theta(t)$$

Advanced

$$G_{AB}^{Adv}\,(t) = -i\, < 0|\hat{A}(t)\,\hat{B}(0) - \hat{B}(0)\,\hat{A}(t)|0 > \; \theta(-t)$$

Keldysh

$$G_{AB}^{K}\,(t) = i\, < 0|\hat{A}(t)\,\hat{B}(0) + \hat{B}(0)\,\hat{A}(t)|0 >$$

Show that for the ground state at $T = 0$, the Fourier transforms of these correlation functions are related by

$$G_{AB}^{K}(\omega) = sgn(\omega)\,(G_{AB}^{Ret}(\omega) - G_{AB}^{Adv}(\omega))$$

Hint: use the spectral (Lehman) representation.

Problem 3

In the lectures we formulated the Random Phase Approximation (RPA) as renormalization of the effective Coulomb interaction

$$
\begin{aligned}
V_{RPA}(q) &= V_0(\vec{q}) - V_0(\vec{q})\mathcal{P}_0(q)V_{RPA}(q) \\
\mathcal{P}_0(q) &= 2i\int \frac{d^4p}{(2\pi)^2} G_0(p+q/2)G_0(p-q/2) = -\chi_0(q)
\end{aligned}
$$

Here $q = (q_0, \vec{q})$, see page 232 of the notes for details. The RPA can also be formulated as renormalization of the charge susceptibility

$$
\begin{aligned}
\mathcal{P}(q) &= i\int e^{iq_0 t}\langle 0|\, T\left\{\rho_{\vec{q}}(t)\rho^{\dagger}_{\vec{q}}(0)\right\}|0\rangle \\
\rho^{\dagger}_{\vec{q}} &= \sum_{\vec{p}\sigma} c^{\dagger}_{\vec{p}+\vec{q}/2,\sigma}\, c_{\vec{p}-\vec{q}/2,\sigma}
\end{aligned}
$$

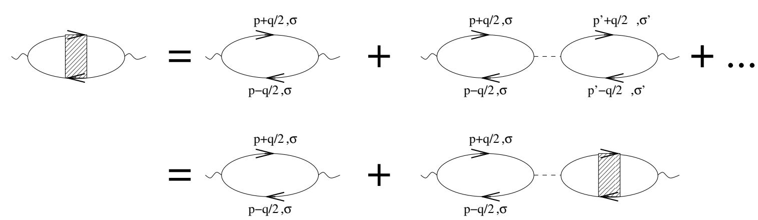

FIG. 1: Problem 3

$$
\mathcal{P}_{RPA}(q) = \mathcal{P}_0(q) - \mathcal{P}_0(q)\cdot V(\vec{q})\cdot \mathcal{P}_{RPA}(q)
$$

which gives

$$
\mathcal{P}_{RPA}(q) = \frac{\mathcal{P}_0(q)}{1 + V(\vec{q})\cdot\mathcal{P}_0(q)}
$$

The last equation is equivalent to

$$
\chi_{RPA}(q) = \frac{\chi_0(q)}{1 - V(\vec{q})\cdot\chi_0(q)}
$$

discussed on page 58 of the notes. Poles of $\mathcal{P}_{RPA}(q)$ describe collective excitations in the particle-hole channel: plasma oscillations (the fluctuating field is $\rho_{\vec{q}}$).

We can generalize the RPA discussion to other susceptibilities, in particular the singlet pairing susceptibility

$$
\begin{aligned}
\mathcal{T}(q) &= i\int e^{iq_0 t}\langle 0|\, T\left\{\Delta_{\vec{q}}(t)\Delta^{\dagger}_{\vec{q}}(0)\right\}|0\rangle \\
\Delta^{\dagger}_{\vec{q}} &= \sum_{\vec{p}} c^{\dagger}_{\vec{p}+\vec{q}/2\uparrow}\, c^{\dagger}_{-\vec{p}+\vec{q}/2\downarrow}
\end{aligned}
$$

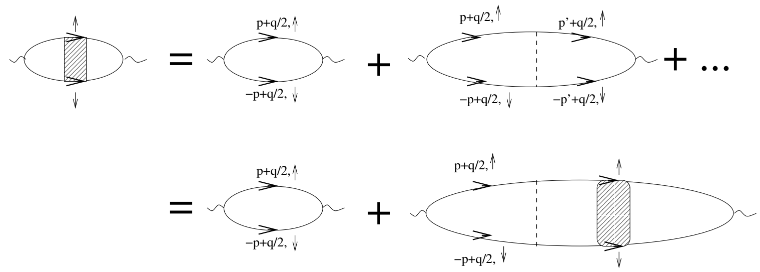

FIG. 2: Problem 3

a) Assuming that the interaction is wavevector independent $V(\vec{q}) = U_0$ show that:

$$\mathcal{T}_{RPA}(q) \;=\; \mathcal{T}_0(q) \;+\; \mathcal{T}_0(q)\; U_0\; \mathcal{T}_{RPA}(q)$$

with

$$\mathcal{T}_0(q) \;=\; i \int \frac{d^4p}{(2\pi)^4}\, G_0(p+q/2)\, G_0\,(-p+q/2)$$

b) Poles of

$$\mathcal{T}_{RPA}(q) \;=\; \frac{\mathcal{T}_0(q)}{1 - U_0\, \mathcal{T}_0(q)}$$

describe collective excitations in the particle-particle channel: The fluctuating field id $\Delta^\dagger_{\vec{q}}$. For the lattice Hamiltonian

$$H \;=\; -t \sum_{<ij>\sigma} c^+_{i\sigma} c_{j\sigma} - \mu \sum_{i\sigma} n_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

show that there is a collective excitation at wavevector $\vec{Q} = (\pi, \pi)$ with energy $\omega_0 = -2\mu + U(1-n)$, where n is the electron filling (half-filling corresponds to $n = 1$). This is the η-excitation discussed in Problem Set 3, Problem 3: $\eta^\dagger = \sum_i (-)^i\, c^+_{i\uparrow}\, c^+_{i\downarrow} = \Delta^\dagger_{\vec{Q}}$. It may be used to understand the interplay of superconductivity and charge density wave order in systems with strong electron-phonon coupling ($BaBi_xPb_{1-x}O_3$, $PbMoS_8$, $A-15$ compounds, etc.; see e.g. Rev.Mod.Phys. 62:113 (1990))

Note, that to get the right energy of the η-mode we need to add the Hartree contribution to the electron self-energy

$$\begin{aligned} \frac{1}{G(p_0,\vec{p})} \;&=\; \frac{1}{G_0(p_0,\vec{p})} - \Sigma \\ \Sigma \;&=\; Un/2 \end{aligned} \qquad (1)$$

c*) Generalize part b) to the triplet pairing channel

$$\mathcal{W}(q) \;=\; i\int e^{iq_0 t}\langle 0|\,T\left\{\pi_{\vec q}(t)\pi^{\dagger}_{\vec q}(0)\right\}|0\rangle$$
$$\pi^{\dagger}_{\vec q} \;=\; \sum_{\vec p} g_p c^{\dagger}_{\vec p+\vec q/2\uparrow}\, c^{\dagger}_{-\vec p+\vec q/2\uparrow}$$

Assume separable interactions $V(\vec q,\vec p,\vec k) = V_0\, g_{\vec p} g_{\vec k}$ (see Figure).

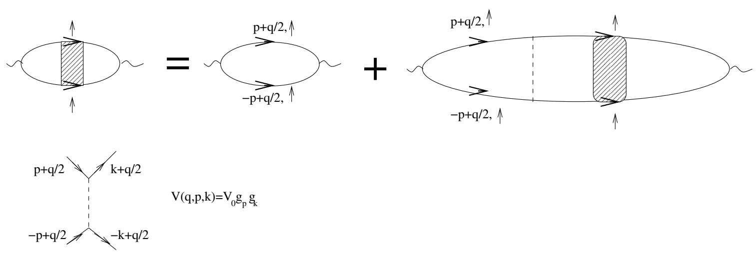

FIG. 3: Problem 3

$$\mathcal{W}(q) \;=\; \mathcal{W}_0(q) \;+\; \mathcal{W}_0(q)\;\; V_0\;\mathcal{W}(q)$$

with

$$\mathcal{W}_0(q) \;=\; i\int \frac{d^4p}{(2\pi)^4}\;\, g^2_{\vec p}\, G_0(p+q/2)\, G_0\,(-p+q/2)$$

d*) Show that the lattice Hamiltonian

$$H \;=\; -t\sum_{<ij>\sigma} c^{+}_{i\sigma}\, c_{j\sigma} \;-\; \mu\sum_{i\sigma} n_{i\sigma} \;+\; J\sum_{<ij>} \vec S_i\, \vec S_j$$

supports the spin tripliet excitations in the particle-particle channel at wavevector (π,π) (the π-excitation), with energy $\omega_0 \;\approx\; -2\mu \;+\; \frac{J}{2}(1-n)$, where n is the filling factor. This π-excitations was proposed as the generator of the SO(5) symmetry in the high T_c cuprates (see Science 275:1089, 1997).

Problem 4

A recently discovered high temperature superconductor MgB_2 (Nature, 410, 63, (01)) has two bands crossing the Fermi surface. It is now believed that superconductivity in this material comes from the usual electron-phonon coupling. A good starting point is then a BCS type two-band Hamiltonian

$$\mathcal{H} = \sum \varepsilon^{(1)}_{k\sigma}\, a^+_{k\sigma}\, a_{k\sigma} + \sum \varepsilon^{(2)}_{q\sigma}\, b^+_{q\sigma}\, b_{q\sigma}$$

$$- \sum_{kk'} V_{kk'}\, a^+_{k\uparrow}\, a^+_{-k\downarrow}\, a_{-k'\downarrow}\, a_{k'\uparrow}$$

$$- \sum_{qq'} U_{qq'}\, b^+_{q\uparrow}\, b^+_{-q\downarrow}\, b_{-q'\downarrow}\, b_{q'\uparrow}$$

$$- \sum_{kq} J_{kq}\, a^+_{k\uparrow}\, a^+_{-k\downarrow}\, b_{-q\downarrow}\, b_{q\uparrow} + h.c.$$

$$with\ \ V_{kk'} = V,\ if\ \ |\xi^{(1)}_k|\,,\ |\xi^{(1)}_{k'}| < \omega_{Dg} \ = \ 0\ otherwise$$

$$U_{qq'} = U,\ if\ \ |\xi^{(2)}_q|\,,\ |\xi^{(2)}_{q'}| < \omega_{Dg} \ = \ 0\ otherwise$$

$$J_{kq} = J,\ if\ \ |\xi^{(1)}_k|\,,\ |\xi^{(2)}_{q}| < \omega_{Dg} \ = \ 0\ otherwise$$

a) Derive the BCS gap equations for this model.

b) Discuss whether there is one or two SC transitions (as a function of temper.) in this system, when the coupling constants V,U, and J are very different (experimentally $U \simeq J \simeq \frac{1}{3}V$).

Problem 5

Find the spin susceptibility of the electrons in an s-wave superconductor at low temperatures ($T << T_c$). Explain your result.

Hint: Write the Hamiltonian using the Bogoliubov operators $\gamma_{k\sigma}$.