Problem 1

Consider an anisotropic XY model

\[ \mathcal{H} = -J_\parallel \sum_{<ij>\parallel} \{ \cos(\theta_{ij}) + \delta \cos(2\theta_{ij}) \} - J_\perp \sum_{<ij>\perp} \cos(\theta_{ij}) \]

where \(<ij>\parallel\) denotes nearest neighbors within a plane, and \(<ij>\perp\) denotes nearest neighbors between planes. In the ordered state, the free energy associated with time independent long wavelength deformations of the phase must be of the form

\[ V_{\text{phase}} = \frac{\gamma_\parallel}{2} \int d^3r (\nabla_\parallel^\theta )^2 + \frac{\gamma_\perp}{2} \int d^3r (\nabla_\perp^\theta )^2 \]

where \(\gamma_\parallel\) and \(\gamma_\perp\) are superfluid stiffnesses within a plane and perpendicular to the planes, respectively.

a) Calculate \(\gamma_\parallel\) and \(\gamma_\perp\) at \(T = 0\).

b) Using linear spin wave theory, find the low temperature expansion of \(\gamma_\parallel(T)\).

Problem 2

Consider a system of hard-core bosons on a two-dimensional lattice

\[ \mathcal{H} = -J \sum_{<ij>} b_i^+ b_j - \mu \sum_i b_i^+ b_i \quad (1) \]

Any site may contain 0 or 1 bosons. Simple variational wavefunctions for this Hamiltonian can be written as

\[ |\psi> = \prod_i (\cos(\theta) + \sin(\theta)b_i^+) |0> \quad (2) \]

where \(|0>\) is an empty lattice.

a) Determine the phase diagram of (1) using variational wavefunctions (2).

b) Discuss the behavior of the superfluid stiffness in the phase diagram.

c) Generalization of (1) to a two-component mixture can be written as

\[ \mathcal{H} = -J \sum_{<ij>} b_i^+ b_j - J \sum_{<ij>} a_i^+ a_j - \mu_1 \sum_i b_i^+ b_i - \mu_2 \sum_i a_i^+ a_i + V \sum_i a_i^+ a_i b_i^+ b_i \quad (3) \]

Here \(n_a \leq 1\) and \(n_b \leq 1\). Assuming \(V > 0\), determine the phase diagram of (3).