Problem 1
Consider an $SO(5)$ quantum rotor model

$$\mathcal{H} = \frac{1}{2\chi} \sum_{i,ab} L_{ab}^2(i) - J \sum_{i<j>a} n_a(i)n_a(j) + g \sum_i (n_1^2 + n_2^2 - n_2^2 - n_3^2 - n_4^2)$$

(1)

Here $n_a(i)$ describes the $SO(5)$ superspins defined for each lattice site $i$ and $L_{ab}(i)$ are operators of the $SO(5)$ rotations. The operators obey the commutation relations

$$[L_{ab}(i), L_{cd}(j)] = -i\delta_{ij}(\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} - \delta_{bc} L_{ad})$$

$$[L_{ab}(i), n_c(j)] = -i\delta_{ij}(\delta_{ac} n_b - \delta_{bc} n_a)$$

$$[n_a(i), n_b(j)] = 0$$

a) Discuss the phase diagram of (1).

b) Calculate the excitation spectrum of (1) in various phases. Hint: write Heisenberg equations of motion for $L_{ab}(i)$, $n_{ab}(i)$ and linearize them.

Problem 2
Consider the 1-dimensional electron system in Fig 1. Let $a_{\pm, k}^+$ be creation operators for the right/left moving electrons of momentum $\pm k_f + k$. We define total spin, total charge, and $\theta$ operators

$$S_\alpha = \frac{1}{2} \sum_{r, ks,s'} a_{r, ks}^+ \sigma^\alpha_{ss'} a_{r, ks'}$$

$$Q = \frac{1}{2} \sum_{ks} (a_{+, ks}^+ a_{+, ks} + a_{-, ks}^+ a_{-, ks} - 1)$$

$$\theta^+ = \sum_k (a_{+, k^+}^+ a_{+, k^-}^- - a_{-, k^+}^+ a_{-, k^-}^-)$$

We also introduce order parameters for triplet superconductivity and antiferromagnetism

$$\psi_{\alpha}^+ = \frac{1}{2} \sum_{k, ss'} a_{+, ks}^+ (\sigma_2 \sigma^\alpha_{ss'}) a_{-, ks'}^+$$

$$N_{\alpha} = \frac{1}{2} \sum_{k, ss'} (a_{+, ks}^+ \sigma^\alpha_{ss} a_{-, ks'}^+ + a_{-, ks}^+ \sigma^\alpha_{ss'} a_{+, ks'})$$

a) Show that spin operators $S_\alpha$ obey the $SO(3)$ spin algebra
\[ [S_\alpha, S_\beta] = i \varepsilon_{\alpha\beta\gamma} S_\gamma \]

b) Show that one can combine operators \( \theta^\dagger, \theta \) and \( Q \) in such a way that they form an \( SO(3) \) isospin algebra

\[ [I_a, I_b] = i \varepsilon_{abc} I_c \]

c*) Show that one can combine real vectors \( N_\alpha, \text{Re} \psi_\alpha, \text{Im} \psi_\alpha \) in such a way that we have an order parameter which transforms as a vector representation under both spin and isospin \( SO(3) \) symmetries. Hint: Vector representation corresponds to \( [O_a, V_b] = i \varepsilon_{abc} V_c \), where \( 0_a \) is a symmetry generator and \( V_b \) is an order parameter.