Problem 1

Consider an $SO(5)$ quantum rotor model

$$\mathcal{H} = \frac{1}{2\chi}\sum_{i,ab} L_{ab}^2(i) - J\sum_{<ij>a} n_a(i)n_a(j) + g\sum_i (n_1^2 + n_5^2 - n_2^2 - n_3^2 - n_4^2)(i) \quad (1)$$

Here $n_a(i)$ describes the $SO(5)$ superspins defined for each lattice site i and $L_{ab}(i)$ are operators of the $SO(5)$ rotations. The operators obey the commutation relations

$$[L_{ab}(i), L_{cd}(j)] = -i\delta_{ij}(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad})$$

$$[L_{ab}(i), n_c(j)] = -i\delta_{ij}(\delta_{ac}n_b - \delta_{bc}n_a)$$

$$[n_a(i), n_b(j)] = 0$$

a) Discuss the phase diagram of (1).

b) Calculate the excitation spectrum of (1) in various phases. Hint: write Heisenberg equations of motion for $L_{ab}(i)$, $n_{ab}(i)$ and linearize them.

Problem 2

Consider the 1-dimensional electron system in Fig 1. Let $a^+_{\pm,\,k}$ be creation operators for the right/left moving electrons of momentum $\pm k_f + k$. We define total spin, total charge, and θ operators

$$S_\alpha = \frac{1}{2}\sum_{r,\,kss'} a^+_{r,\,ks}\,\sigma^\alpha_{ss'}\,a_{r,\,ks'}$$

$$Q = \frac{1}{2}\sum_{ks}(a^+_{+,\,ks}\;a_{+,\,ks} + a^+_{-,\,ks}\,a_{-,\,ks} - 1)$$

$$\theta^+ = \sum_k (a^+_{+,\,k\uparrow}\,a^+_{+,-k\downarrow} - \,a^+_{-,\,k\uparrow}\,a^+_{-,\,-k\downarrow})$$

We also introduce order parameters for triplet superconductivity and antiferromagnetism

$$\psi^+_\alpha = \frac{1}{i}\sum a^+_{+,\,ks}(\sigma_2\,\sigma^\alpha)_{ss'}\;a^+_{-,\,-ks'}$$

$$N_\alpha = \frac{1}{2}\sum_{kss'}(a^+_{+,\,ks}\,\sigma^\alpha_{ss'}\,a_{-,\,ks'} + \,a^+_{-,\,ks}\,\sigma^\alpha_{ss'}\,a_{+,\,ks'})$$

a) Show that spin operators S_α obey the $SO(3)$ spin algebra

$$[S_\alpha, S_\beta] = i\,\varepsilon_{\alpha\beta\gamma} S_\gamma$$

b) Show that one can combine operators θ^+, θ and Q in such a way that they form an $SO(3)$ isospin algebra

$$[I_a, I_b] = i\,\varepsilon_{abc} I_c$$

c*) Show that one can combine real vectors N_α, $Re\,\psi_\alpha$, and $Im\,\psi_\alpha$ in such a way that we have an order parameter which transforms as a vector representation under both spin and isospin $SO(3)$ symmetries. Hint: Vector representation corresponds to $[O_a, V_b] = i\,\varepsilon_{abc} V_c$, where 0_a is a symmetry generator and V_b is an order parameter.